

ALL BOREL GROUP EXTENSIONS OF FINITE-DIMENSIONAL REAL SPACE ARE TRIVIAL

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ABSTRACT. For $N \geq 2$, we study the structure of definable abelian group extensions of the additive group $(\mathbb{R}^N, +)$ by countable abelian (Borel) groups G . Given an extension H of $(\mathbb{R}^N, +)$ by G , we measure the definability of H by investigating its complexity as a Borel set. We do this by combining homological algebra and descriptive set theory, and hence study the Borel complexity of those functions inducing H , the abelian cocycles. We prove that, for every $N \geq 2$, there are no non-trivial Borel definable abelian cocycles coding group extensions of $(\mathbb{R}^N, +)$ by a countable abelian group G , and hence show that no non-trivial such group extensions exist. This completes the picture first investigated by Kanovei and Reeken in 2000, who proved the case $N = 1$, and whose techniques we adapt in this work.

1. INTRODUCTION

In this paper, we contribute to the study of *definable classical mathematics*. By classical mathematics we mean those mathematical subjects traditionally unrelated to mathematical logic. The idea of definability is interpreted broadly: as opposed to the universal yet rigid model-theoretical definition, we mean the classification of instances of classical mathematical problems in terms of measures at home in mathematical logic.

One of the earlier examples of the application of logic to classical mathematics was given by Shelah's celebrated solution to the Whitehead problem [36], but many have followed suit. Applications of this research programme feature prominently in major areas of contemporary logic such as reverse mathematics [11, 37] and computability theory [38]. More finely, the study of reductions and effective measures of complexity play an important role throughout modern logic, emphasised by notions such as Weihrauch reducibility [4], computable analysis [41], algorithmic randomness [25, 32, 9], and many more. Crucially, many of these admit natural connections to classical mathematics, such as the effective relationship between fractal geometry and algorithmic randomness [29, 27, 15, 28].

In this paper, we continue this trend. We choose as our measure framework the study of subsets of the reals in terms of descriptive set theory and apply it to (homological) algebra: we consider the problem of group extensions of abelian groups by abelian groups, and measure which of them are *definable* in the context of descriptive set theory. Analysing the *Borel definability* of group extensions garnered interest in the 1960s through work of Moore [30] and DuPre [10], and was re-investigated by Kanovei and Reeken in the early 2000s [20, 22, 21]. Instead of taking a strongly algebraic point of view as Moore and DuPre had done, their work was motivated by ideas

Date: 15 November, 2024.

2020 Mathematics Subject Classification. 54H05 (Primary) 20K35, 20J06, 03E40 (Secondary).

Key words and phrases. Borel definability, descriptive set theory, group extensions, group cohomology, forcing.

in *Hyers-Ulam-Rassias-stability* [16, 39, 40, 33] which measures the stability of spaces (and, more generally, categories) in terms of homomorphisms: can every *almost homomorphism* be closely approximated by a *strict* homomorphism? For certain groups, group extensions stand in equivalence with maps called *cocycles*, whose properties can be likened to Hyers-Ulam-Rassias-(in)stability. In combination with the measurability tools of Polish spaces, *definable* group extensions can be carefully analysed.

This analysis is our motivation for this paper. We generalise a result of Kanovei and Reeken concerning the existence of definable group extensions [22] to higher-dimensional topological spaces, hence proving that the definable algebraic structure (in the sense of Borel measurability) of abelian group extensions of the additive groups $(\mathbb{R}, +)$ and $(\mathbb{R}^N, +)$ for $N > 1$ does not differ.

To formalise this, let A, G be groups. A **group extension** of A by G is a *short exact sequence*: a sequence c of the form

$$c: 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{j} A \longrightarrow 0$$

where i, j are group homomorphisms, i is injective, j is surjective, and where $\text{im}(i) = \ker(j)$. Two group extensions c, c' with

$$c': 0 \longrightarrow G \xrightarrow{i'} E' \xrightarrow{j'} A \longrightarrow 0$$

are *equivalent* if there is a group isomorphism p for which the following diagram commutes:

$$\begin{array}{ccccccc} & & & E & & & \\ & & i \nearrow & \downarrow p & \searrow j & & \\ 0 & \longrightarrow & G & & A & \longrightarrow & 0 \\ & & i' \searrow & \downarrow & \nearrow j' & & \\ & & & E' & & & \end{array}$$

If we require that all groups G, E, A are abelian, then every extension of A by G is in fact induced by an *abelian cocycle* (or *factor set*)—a function from A^2 to G which encodes the structure of E —and vice versa. This is how we study the structure of group extensions here: we consider group extensions from the point of view of abelian cocycles and their properties in terms of functional equations; their structure is captured by the *cohomology group*.

The identification of group extensions in terms of functions between groups has spurred the investigation of definable group extensions: those induced by an abelian cocycle which is *definable*. In this paper, *definable* means *Borel measurable*: we consider groups which carry a natural Polish structure, and investigate those group extensions which are induced by Borel cocycles.

Definition 1.1. A group G is a **Borel group** if G is a Borel subset of a standard Borel (or Polish) space, and the group operation is a **Borel map**: the pre-image of every Borel set is Borel in the inherited topology.

Suppose G, E, A are abelian Borel groups inside standard Borel spaces. A group extension $c: 0 \rightarrow G \rightarrow E \rightarrow A \rightarrow 0$ is **Borel definable** (or just **Borel**) if the associated abelian cocycle generating E is a Borel map.

In order to take advantage of the identification between group extensions and cocycles, **we assume from now on that all groups are abelian**. Further, **the quotient G is always assumed to be countable**.

With these conventions established, the following theorem [22, Theorem 49] classifies, up to isomorphism between group extensions, all definable group extensions of the additive group $(\mathbb{R}, +)$ by a (countable) group G in the Borel framework.

Theorem 1.2 (Kanovei and Reeken, 2000). *For every group G , the only Borel definable group extension of $(\mathbb{R}, +)$ by G is the trivial extension. In other words, every Borel cocycle $C: (\mathbb{R}^2)^2 \rightarrow G$ is a Borel coboundary.*

In sections 3 and 4, we prove the following theorem, extending Kanovei and Reeken's result.

Theorem 4.6. For every $N < \omega$ and every group G , the only Borel definable group extension of $(\mathbb{R}^N, +)$ by G is the trivial extension.

The proof uses geometrical arguments, which on the one hand are harder to prove in the multidimensional case as opposed to the case \mathbb{R} in [22], while on the other hand translate throughout the higher-dimensional cases. As general ideas of the proof of theorem 4.6 are most easily explained in the case $N = 2$, we prove the following result first:

Theorem 3.21. For every group G , the only Borel definable group extension of $(\mathbb{R}^2, +)$ by G is the trivial extension.

1.1. The structure of this paper. In section 2 we outline the basics of group theory and the theory of group extensions needed for our arguments. We also recall the fundamentals of descriptive set theory which we later use to classify the degree of definability of group extensions.

As we will be using forcing arguments, we shall also consider its basic notions. However, these will not concern extending models of set theory, and hence remain gentle.

In sections 3 and 4, we prove theorems 3.21 and 4.6, respectively. The proof of theorem 3.21 follows structurally the analogous argument in [22], yet requires more sophisticated arguments to overcome the geometrical differences between \mathbb{R} and \mathbb{R}^2 . The lemmas we prove, however, can then be extended to prove the theorem on $(\mathbb{R}^N, +)$ for any $N > 2$. We show how to do this in section 4, where we also conclude theorem 4.6.

We close with section 5, in which we state various open questions, and where we outline related avenues for research.

2. BACKGROUND

The theory of group extensions of abelian groups by abelian groups can be expressed in two ways: *category-theoretically* (i.e. in terms of short exact sequences, as indicated in section 1) and in terms of *group cohomology*, which captures the relationship between certain functions between groups: *cocycles* and *coboundaries*. The former approach has lead to the study of the **Ext** functor [12], which is of course of independent interest. In this paper, however, we take the latter approach. We do this since: if all groups concerned have a Borel structure, then the Borel complexity of cocycles—which are themselves functions between groups—can be measured and hence analysed.

Let A be a Borel group. A function $C: A^2 \rightarrow G$ is an **abelian 2-cocycle** (or just **abelian cocycle**) if $C(x, 0) = 0$ and

$$C(x, y) = C(y, x) \quad \text{and} \quad C(x, y) + C(x + y, z) = C(x, z) + C(x + z, y)$$

for all $x, y, z \in A$. An abelian cocycle is a **coboundary** if there exists a map $h: A \rightarrow G$ such that

$$C(x, y) = C_h(x, y) = h(x) + h(y) - h(x + y).$$

The quotient group of cocycles modulo coboundaries yields the **cohomology group** $H^2(A, G)$ [35, 9.2], which measures complexity: the more complicated $H^2(A, G)$ is as a group, the more complicated is the structure of group extensions of A by G . It is clear that two cocycles are **equivalent** if their difference is a coboundary¹.

Remark. The definition of coboundaries hints at the relationship between the theory of group extensions and the aforementioned Hyers-Ulam-Rassias-stability framework of almost homomorphisms: in the sense of Ulam, coboundaries are trivial objects (well-behaved almost homomorphisms), just as they are trivial objects in the study of group cohomology (by definition of the quotient group).

To explain our theorems, we need to introduce descriptive set theory. A topological space is **Polish** if it is separable and completely metrisable. The *definable* subsets of an uncountable Polish space X can be stratified in hierarchies. At the lowest level sits the **Borel hierarchy**, whose sets are generated by the σ -algebra containing the open subsets of X . A set is **Borel** if it appears in this hierarchy of length ω_1 . A function $F: X \rightarrow Y$ between Polish spaces is a **Borel function** if the pre-image of every Borel set is Borel. Hence, every continuous function is Borel. Further, a function between Polish spaces is Borel if and only if its graph is *analytic*, i.e. the continuous image of a Borel set [23, II.14.12]. Since Polish spaces are separable and Hausdorff, they have a countable basis of **basic open sets**, and so every Borel set A has a **Borel code**: an element of ω^ω which codes the construction of A from the basic open sets and countably many operations of countable union and complementation. For details, see [23, 19].

A simple method to determine whether a set of reals is Borel is provided by the fundamental relationship between descriptive set theory and computability theory (see references [31, 13, 5]). The following folklore result is emblematic of this relationship. Some basic concepts follow: if Φ is a computable procedure (e.g. a Turing machine) and x is a real (in \mathbb{R} , 2^ω , $\omega^\omega \dots$), let $\Phi(x^{(\alpha)})$ denote the procedure Φ initiated with oracle access to the α -th jump of x . If $x, z \in 2^\omega$ then $x \oplus z$ is the join of x and z . (The join of elements of \mathbb{R} is usually expressed as the join of their binary expansions. For classical computability theory, see e.g. [38].)

Proposition 2.1 (cf. [8, 34]). *Let X be uncountable Polish. A set $A \subseteq X$ is Borel if there exists a code $z \in \omega^\omega$, an ordinal² $\alpha < \omega_1$, and a computable procedure Φ for which*

$$x \in A \iff \Phi((x \oplus z)^{(\alpha)}) \text{ halts in finite time.}$$

Proof. To elucidate the intuition, we sketch the proof of the case $\alpha = 1$ for $X = \mathbb{R}$. If A is open, write $A = \bigcup (a_n, b_n)$ where $a_i, b_i \in \mathbb{Q}$. This sequence of pairs is coded by some z (via e.g. the Cantor pairing function). For each pair (a_n, b_n) coded in z , ask whether $a_n < x < b_n$. This is a finite operation since each interval (a_n, b_n) is open. \square

¹To see the explicit relationship between cocycles $C: A^2 \rightarrow G$ and group extensions of A by G we recommend in particular Chapter 9 of [12]. There, it is also shown how the notion of equivalence given here agrees with the commutative-diagram-equivalence from section 1.

²To the reader familiar with hyperarithmetical theory: $\alpha < \omega_1^z$, the first ordinal not computable from z .

2.1. Algebraic properties of abelian cocycles. In the subsequent arguments, we generally separate *algebraic properties* of abelian cocycles from *non-algebraic* properties.

As the name suggests, algebraic properties hold for all abelian cocycles; the underlying space and its topology are inconsequential. On the contrary, non-algebraic properties depend on the underlying topological and order-theoretical properties. For instance, the reals \mathbb{R} are ordered, while \mathbb{R}^2 is not. Hence, any proof of theorem 1.2, if adapted to \mathbb{R}^2 , must take care of such structural differences.

The difficulties of extending theorem 1.2 reduce to proving topological lemmas which permit the application of algebraic properties. This is our task in the following section. In this section, we collect useful algebraic properties of abelian cocycles—all can be found in [22]. Fix an uncountable group X and a group G and let $C: X^2 \rightarrow G$ be an abelian cocycle.

Lemma 2.2. *For all $x_1, x_2, x_3 \in X$ we have*

$$C(x_1, x_2) = C(x_1, x_3) + C(x_1 + x_3, x_2) - C(x_1 + x_2, x_3).$$

Cocycles can be extended naturally to accept more arguments: let

$$C(x_1, x_2, x_3) = C(x_1, x_2) + C(x_1 + x_2, x_3)$$

and hence recursively define

$$C(x_1, \dots, x_{n+1}) = C(x_1, \dots, x_n) + C(x_1 + \dots + x_n, x_{n+1}).$$

Various useful identities can be deduced from this recursive definition, such as the following.

Lemma 2.3. *If $x_1, \dots, x_n, x_{n+1}, \dots, x_m \in X$, let $x = x_1 + \dots + x_n$ and $x' = x_{n+1} + \dots + x_m$. Then*

$$C(x_1, \dots, x_m) = C(x_1, \dots, x_n) + C(x_{n+1}, \dots, x_m) + C(x, x').$$

Recall that a cocycle C is a coboundary if there exists a map $h: X \rightarrow G$ for which

$$C(x_1, x_2) = h(x_1) + h(x_2) - h(x_1 + x_2).$$

A simple induction proves:

Lemma 2.4. $C_h(x_1, \dots, x_n) = h(x_1) + \dots + h(x_n) - h(x_1 + \dots + x_n)$

Further, since $H^2(X, G)$ is a group it immediately follows that:

Lemma 2.5. *If C_h and C_k are coboundaries, then $C_h + C_k = C_{h+k}$, where $(h+k)(x) = h(x) + k(x)$. In particular, if C_h and C_k are Borel coboundaries (i.e. h and k are Borel maps), then so is C_{h+k} .*

2.2. The set-theoretical setup. We work in a set-theoretical universe V satisfying ZFC. For a sufficiently rich finite fragment ZFC^* of ZFC, consider a countable transitive set model $M \models \text{ZFC}^*$. Meta-theoretically, our arguments proceed as follows: with a Borel cocycle C and M fixed, we work inside V and argue that C is in fact a Borel coboundary. We often allude to the fact that truth in a sufficiently rich extension $M[G]$ of M is forced by some condition (read open set) in M : there is a comeagre set in V on which the function is constant, for instance. For suitable G , this gives us a result on V , hence proving that C enjoys said properties *in the real world*. The fact that Borel functions have codes whose interpretations are absolute between transitive models is crucial for this argument to work.

As is customary, when we write “let M be a countable transitive model of ZFC” we mean M to satisfy a finite fragment of ZFC sufficiently large for the arguments at hand.

In view of proving theorem 3.21, we now consider the specific case $A = (\mathbb{R}^2, +)$. We also write \mathbb{R}^2 in place of $(\mathbb{R}^2, +)$ to save time.

While the following set-theoretical setup is expressed in terms of \mathbb{R}^2 , it should be clear that it in fact holds for \mathbb{R}^N for all $N < \omega$. We work in V and fix a countable group G . Suppose $C: (\mathbb{R}^2)^2 \rightarrow G$ is a Borel function. Let $z \in \omega^\omega$ be a Borel code for C . Now consider a fixed countable transitive model M of ZFC such that $\{z, G\} \subset M$.

Let $\mathbb{P}_2 = \{(q, q') \times (r, r') \mid q, q', r, r' \in \mathbb{Q}\}$, our version of Cohen-forcing for pairs. As always, $p = I \times I' \leq q = J \times J'$ (i.e. p is stronger than q) if and only if $I \subseteq J$ and $I' \subseteq J'$. Further, define \mathbb{P} to be the classical Cohen forcing whose conditions are open sets (q, r) for $q, r \in \mathbb{Q}$. Then $\mathbb{P} \times \mathbb{P} = \mathbb{P}_2$ as sets, and also as forcing notions, as the following classical theorem shows.

Theorem 2.6 ([24]). *A filter $G_0 \times G_1$ is M -generic for \mathbb{P}_2 iff one of the following holds:*

- (1) G_0 is M -generic for \mathbb{P} and G_1 is $M[G_0]$ -generic for \mathbb{P} .
- (2) G_1 is M -generic for \mathbb{P} and G_0 is $M[G_1]$ -generic for \mathbb{P} .

Corollary 2.7. *Forcing with $\mathbb{P}_2 \times \mathbb{P}_2$ is equivalent to forcing with the product $\mathbb{P} \times \mathbb{P} \times \mathbb{P} \times \mathbb{P}$, which is equivalent to iterating classical Cohen forcing four times over M .*

After forcing with $\mathbb{P}_2 \times \mathbb{P}_2$, we denote the resultant M -generic tuple by $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, where theorem 2.7 implies that α_0 is M -generic, α_1 is $M[\alpha_0]$ -generic, α_2 is $M[\alpha_0, \alpha_1]$ -generic, and so forth.

Lemma 2.8. *There exist $p = I \times J$ and $q = I' \times J'$ and $g \in G$ such that $\text{diam}(q) > \text{diam}(p)$, I lies in the first quadrant, and if $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in p \times q$ is M -generic then $C((\alpha_0, \alpha_1), (\alpha_2, \alpha_3)) = g$.*

In other words, some condition forces C to be constant on all generics inside it. For notation, we say $D(I \times J) < D(I' \times J')$ if $I \subsetneq I'$ and $J \subsetneq J'$.

Proof. Firstly, note that being a Borel code is Π_1^1 [18, II.25.44], and hence absolute by Shoenfield absoluteness [18, II.25.20]. Therefore, the interpretation C^M of z in M is a total function from \mathbb{R}^M to G (recalling that $G = \{g_i \mid i < \omega\} \in M$), and this fact is forced by the empty condition. Hence we may assume p extends some condition in the first quadrant.

Let $(\dot{\alpha}_0, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3)$ be a name for the M -generic. By the fundamental theorem of forcing, there exists a condition $p \Vdash C(\dot{\alpha}_0, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3) = \check{g}_i$ for some $i < \omega$. Write $p = I \times J \times I' \times J'$. Of course, we can assume $D(I \times J)$ to be smaller than $D(I' \times J')$. By theorem 2.7, this suffices. \square

From now on, with M fixed, we introduce the following notation.

Definition 2.9. *If $p \in \mathbb{P}$ let p^* denote the set (in V) of M -generics contained in the condition p . We also relativise: if x is a real then $p^{*[x]}$ denotes the set of $M[x]$ -generics contained in p .*

Remark. A real is M -generic if and only if it intersects every dense open subset of \mathbb{P} that is an element of M . Since every open set in \mathbb{P} (and hence in \mathbb{P}_2 , etc.) has a Borel code (since \mathbb{R}^n is Polish), the following observation is immediate: since \mathbb{R} is Baire,

the class of M -generics for \mathbb{P} is a comeagre subset of \mathbb{R} *as viewed in* V . Hence, every condition p contains a comeagre-in- p set of M -generics.

3. THE CASE $N = 2$

In this section, we prove that every Borel definable group extension of $(\mathbb{R}^2, +)$ by a group G can be trivialised by a Borel coboundary. Let

$$C: (\mathbb{R}^2)^2 \rightarrow G$$

be a Borel cocycle. We argue in a sequence of lemmas which eventually reduces C to a sum of Borel coboundaries.

Initially, we follow closely the structure laid out in [22], and the lemmas of algebraic type carry over from their work. As an example, the base case in theorem 3.1 is virtually identical to their arguments. However, the lemmas needed for our 2-dimensional (and beyond) proof, viz. theorem 3.2 and theorem 3.3, are more technical due to the topological and geometrical differences between \mathbb{R} and \mathbb{R}^2 . This is particularly emphasised in section 4, where we prove the general case $N > 2$.

3.1. Geometrical complications. In the process of trivialising the Borel cocycle C , it is our goal to show that C is well-behaved on large and well-behaved (e.g. closed) subsets of \mathbb{R}^2 . We usually argue via generics (as there is a large—in fact comeagre—set of them in V) and show that C is well-behaved on generics first; then we reduce to non-generics.

The following theorem 3.1—an adaptation of Lemma 52 in [22]—is the first step in this reduction process. We break it down into two cases:

Lemma 3.1. *Suppose $n \in \{2, 3\}$, and assume that $x_1, y_1, \dots, x_n, y_n \in p^*$ satisfy*

$$\sum_{i \leq n} x_i = \sum_{i \leq n} y_i.$$

Then $C(x_1, \dots, x_n) = C(y_1, \dots, y_n)$.

The proof is very similar to that of [22]; however, in our context, it requires the following geometrical lemma (which is trivial on \mathbb{R} and hence was not required in [22]).

Lemma 3.2. *For $x_1, y_1, x_2, y_2, x_3, y_3 \in p^*$ satisfying $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$, there exists a non-zero $\epsilon \in \mathbb{R}^2$ for which*

$$\{x_1 + \epsilon, x_2 - \epsilon, y_2 + \delta\} \subset p$$

where $\delta = (y_1 - x_1) - \epsilon$. This ϵ can be chosen to be $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ -generic over M .

Let π_0 denote the projection function onto the first coordinate in \mathbb{R}^2 , and define π_1 similarly.

Proof. Suppose we are given $x_1, y_1, x_2, y_2, x_3, y_3 \in p^*$. First, assume that

$$\pi_0(x_1) = \pi_0(y_1) \quad \text{and} \quad \pi_1(x_1) > \pi_1(y_1)$$

where the choice of indices and coordinates is made without loss of generality. There are two cases.

- (i) First, assume the two remaining pairs can be split up into (x_2, y_3) and (x_3, y_2) both not pointing downwards, and let $D_1 = \pi_1(x_1) - \pi_1(y_1)$. Since $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$, we may assume w.l.o.g. that $\pi_1(y_2) - \pi_1(x_3) \geq D_1/2$. Let $\epsilon = (0, \pi_1(x_2) - \pi_1(y_3) - \gamma)$ where $\gamma > 0$ is sufficiently small and generic (such a γ exists since p is open). It is easily verified that this ϵ works.
- (ii) Second, suppose only one of the two remaining pairs points upwards, w.l.o.g. (x_2, y_2) ; hence $\pi_1(y_2) - \pi_1(x_2) \geq D_1$. Then choose $\epsilon = (0, -\gamma)$ where $0 < \gamma < D_1/2$ is sufficiently generic. Again, it is easily verified that this ϵ works.

Now suppose that no pair (x_i, y_j) out of $x_1, y_1, x_2, y_2, x_3, y_3$ can be arranged to share a coordinate. We argue geometrically: since $\sum_{i \leq 3} x_i = \sum_{i \leq 3} y_i$, we know that $\sum_{i \leq 3} (y_i - x_i) = 0$; hence consider the triangle Δ whose sides are given by the vectors $(x_i - y_i)$. Its set of vertices V is identified with the pairs of vectors that induce the edges. In particular, let v be the vertex of Δ where $(y_2 - x_2)$ and $(y_1 - x_1)$ meet.

Let R_i be the unique rectangle with diagonal $(y_i - x_i)$ and sides parallel to the horizontal and vertical axes; then $\bigcup_{i \leq 3} R_i$ bounds a rectangle inside Δ . Further, each R_i is a shift of the unique condition p_i bounding the pair (x_i, y_i) .

By an easy geometrical observation, and w.l.o.g., we have $R_1 \cap R_2 \neq \emptyset$, and $R_2 \cap R_3 = \emptyset$. Now, let $\epsilon = \beta + \gamma$ where $\beta \in R_1 \cap R_2$, the shift γ is sufficiently generic, and the following hold:

- (1) $x_1 + \epsilon \in p$
- (2) $x_2 - \epsilon \in p$
- (3) $v + \epsilon \in R_3$

A sufficiently small γ exists since p is open. By symmetry of R_3 (and hence of p_3), since $v + \epsilon \in R_3$, we have $y_3 + \delta \in p_3 \subset p$. Relabelling y_3 to y_2 completes the proof. \square

Proof of theorem 3.1. For $n = 2$, the proof is identical to the argument in [22], which we include here for completeness. Let $x_1, x_2, y_1, y_2 \in p^*$ satisfy $x_1 + x_2 = y_1 + y_2$. Take a sufficiently small condition $q < p$ such that there exists a generic $z \in q^{*[x_1, x_2, y_1, y_2]}$ for which $z' = z + (y_1 - x_1) \in q$. This exists since p is open. Now $C(x_1, x_2, z, z') = C(y_1, y_2, z, z')$ by the following argument: by the properties of abelian cocycles,

$$\begin{aligned} C(x_1, x_2, z, z') &= C(x_1, z') + C(x_2, z) + C(x_1 + z', x_2 + z) \\ &= 2g + C(x_1 + z', x_2 + z) \end{aligned}$$

since the pairs in the first two terms are generic (cf. theorem 2.7). Similarly,

$$\begin{aligned} C(y_1, y_2, z, z') &= C(y_1, z) + C(y_2, z') + C(y_1 + z, y_2 + z') \\ &= 2g + C(y_1 + z, y_2 + z'). \end{aligned}$$

Now recall that $x_1 + x_2 = y_1 + y_2$; hence $x_2 + z = y_2 + z'$, and the equality $z' + x_1 = z + y_1$ is immediate from the definition. Further,

$$\begin{aligned} C(x_1, x_2, z, z') &= C(x_1, x_2) + C(z, z') + C(x_1 + x_2, z + z') \\ &= C(y_1, y_2) + C(z, z') + C(y_1 + y_2, z + z') \\ &= C(y_1, y_2, z, z') \end{aligned}$$

from which the case $n = 2$ follows immediately.

For $n = 3$, we use theorem 3.2: let (ϵ, δ) be as provided by the lemma. Then define, as in [22], the sequences (x'_i) and (y'_i) such that $x'_3 = y'_3$ and

$$x'_1 = x_1 + \epsilon, \quad x'_2 = x_2 - \epsilon, \quad y'_1 = y_1 - \delta, \quad y'_2 = y_2 + \delta$$

which yields the result at once from the special case above and the properties of abelian cocycles. \square

We prove below the full version of theorem 3.1. Due to the topological differences between \mathbb{R} and \mathbb{R}^2 , the general case of our proposition below is more involved than Kanovei and Reeken's counterpart.

Proposition 3.3. *Theorem 3.1 holds in fact for all $n \geq 2$.*

We will prove the proposition by induction; consider case the $k + 1$. Before we give the intricate general proof, we reproduce the *special case* from [22], which also applies in the case $N = 2$ (and in fact for every N): we assume that $x_\ell = y_\ell$ for some $\ell \leq k + 1$. To that end, suppose the proposition holds below $k + 1$, let $x_1, y_1, \dots, x_{k+1}, y_{k+1} \in p^*$ be given, and suppose w.l.o.g. that $x_1 = y_1$. Then the lemma follows immediately:

$$\begin{aligned} C(x_1, \dots, x_{k+1}) &= C(x_2, \dots, x_k) + C(x_2 + \dots + x_{k+1}, x_1) \\ &= C(y_2, \dots, y_k) + C(y_2 + \dots + y_{k+1}, y_1) = C(y_1, \dots, y_{k+1}) \end{aligned}$$

by the inductive hypothesis and the definition of C for more than two inputs, as required.

The general proof of theorem 3.3 is explained geometrically by considering the $k + 1$ -gon Γ constructed by concatenating the vectors $(y_1 - x_1), \dots, (y_{k+1} - x_{k+1})$ (recall that the specific labelling of vectors is inconsequential). As before, for every $(y_i - x_i)$, let R_i denote the rectangle bounding it, which can be considered the translation of an associated condition $p_i \subset p$. Since p is open, we may extend p_i by a small amount in all directions without leaving p ; we use this fact repeatedly.

First, we prove another geometrical lemma.

Lemma 3.4. *With $x_1, y_1, \dots, x_k, y_k \in p^*$ and $k \geq 3$, its associated k -gon Γ can be transformed into a k -gon Γ' containing an edge parallel to the second axis without changing the value of C on Γ' .*

Proof. Let Γ be the k -gon induced by the vectors $(y_i - x_i)$. W.l.o.g. let $(y_1 - x_1)$ be the vector with the smallest positive π_0 -shift; that is, $0 < b = \pi_0(y_1) - \pi_0(x_1)$ is minimal among all pairs (x_i, y_i) . Since $\sum x_i = \sum y_i$, there exists $(y_2 - x_2)$ for which $\pi_0(x_2) > \pi_0(y_2)$. Write $b = e + d$ where e is sufficiently generic, and define

$$\begin{aligned} x'_1 &= x_1 + (e, 0), \quad x'_2 = x_2 - (e, 0) \\ y'_1 &= y_1 - (d, 0), \quad y'_2 = y_2 + (d, 0) \end{aligned}$$

and put $x'_i = x_i$ and $y'_i = y_i$ everywhere else. Now, all $x'_i, y'_i \in p^*$ by minimality of b , and further, $\sum x'_i = \sum x_i$ and $\sum y'_i = \sum y_i$. But since $k \geq 3$, there exists j for which $x_j = x'_j$ and $y_j = y'_j$. By the special case,

$$C(x_1, \dots, x_k) = C(x'_1, \dots, x'_k) \quad \text{and} \quad C(y_1, \dots, y_k) = C(y'_1, \dots, y'_k).$$

Finally, the vector $(y'_1 - x'_1)$ is vertical since $\pi_0(x'_1) = \pi_0(x_1) + e = \pi_0(y_1) - d = \pi_0(y'_1)$. \square

By symmetry, the same argument holds for the first axis (or any axis in \mathbb{R}^N). The following proposition can be considered a repeated application of theorem 3.4, which “smooths” the k -gon Γ induced by the set of points $x_1, y_1, \dots, x_{k+1}, y_{k+1}$ until one of its sides vanishes.

Proof of theorem 3.3. Suppose $x_1, y_1, \dots, x_{k+1}, y_{k+1} \in p^*$ and there is no pair (i, j) for which $x_i = y_j$; hence the special case does not apply. By theorem 3.4, we may assume the vector $(y_1 - x_1)$ to be vertical, and assume w.l.o.g. that $\pi_1(x_1) > \pi_1(y_1)$. By minimality, there exists $n < k + 1$ and i_1, \dots, i_n such that

$$\sum_{j \leq n} (\pi_1(y_{i_j}) - \pi_1(x_{i_j})) \geq \pi_1(x_1) - \pi_1(y_1)$$

and $\pi_1(y_{i_j}) - \pi_1(x_{i_j}) > 0$ for all $j \leq n$. We now build sequences $(y_i^{(\ell)} - x_i^{(\ell)})$ and $(e^{(\ell)}, d^{(\ell)})$ for $1 \leq \ell \leq n$ as follows. First, let $x_i^{(1)} = x_i$ and $y_i^{(1)} = y_i$ for all $i \leq k + 1$. We construct the sequence by recursion on ℓ : with $(y_i^{(\ell)} - x_i^{(\ell)})$ given, consider the following two cases:

- (a) Suppose there exist $e, d \in (0, \pi_1(y_{i_\ell}) - \pi_1(x_{i_\ell}) + a)$ (where a is sufficiently small such that $p_{i_\ell} + (0, a) \subset p$) for which e is sufficiently generic and

$$x_1^{(\ell)} - (0, e) = y_1^{(\ell)} + (0, d).$$

Then let $(e^{(\ell)}, d^{(\ell)}) = (e, d)$ and stop the recursion.

- (b) If no such pair exists, pick $e \in (\pi_1(y_{i_\ell}) - \pi_1(x_{i_\ell}), \pi_1(y_{i_\ell}) - \pi_1(x_{i_\ell}) + a)$, sufficiently generic, and put $(e^{(\ell)}, d^{(\ell)}) = (e, e)$.

Now define:

$$\begin{aligned} x_1^{(\ell+1)} &= x_1^{(\ell)} - (0, e^{(\ell)}), \quad x_{i_\ell}^{(\ell+1)} = x_{i_\ell}^{(\ell)} + (0, e^{(\ell)}) \\ y_1^{(\ell+1)} &= y_1^{(\ell)} + (0, d^{(\ell)}), \quad y_{i_\ell}^{(\ell+1)} = y_{i_\ell}^{(\ell)} - (0, d^{(\ell)}) \end{aligned}$$

The following claims ensure that the result follows.

Claim 1. $C(x_1, \dots, x_{k+1}) = C(x_1^{(\ell)}, \dots, x_{k+1}^{(\ell)})$ for all $\ell \leq n$.

Proof of Claim 1. The construction shows that $\sum_i x_i = \sum_i x_i^{(\ell)}$. Since at least one value $x_j^{(\ell)}$ is preserved when passing from $x_j^{(\ell)}$ to $x_j^{(\ell+1)}$, induction proves the result from the special case. The same argument holds for the sequences $(y_i^{(\ell)})$. Since $\sum x_i = \sum y_i$, we have for all ℓ, ℓ'

$$\sum_i x_i^{(\ell)} = \sum_i y_i^{(\ell')}. \quad \dashv$$

Claim 2. Case (a) above occurs in fewer than n steps.

Proof of Claim 2. Let $D^{(1)} = \pi_1(x_1) - \pi_1(y_1)$. Suppose Case (a) does not occur. Then, by construction,

$$e^{(1)} > \pi_1(y_{i_1}) - \pi_1(x_{i_1})$$

and so

$$D^{(2)} = \pi_1(x'_1) - \pi_1(y'_1) < D^{(1)} - e^{(1)}.$$

Since $\sum \pi_1(y_{i_j}) - \pi_1(x_{i_j}) \geq \pi_1(x_1) - \pi_1(y_1)$, the result follows by induction. \dashv

Claim 3. When Case (a) is applied at stage $\ell + 1$, then

$$C(x_1^{(\ell+1)}, \dots, x_{k+1}^{(\ell+1)}) = C(y_1^{(\ell+1)}, \dots, y_{k+1}^{(\ell+1)}).$$

Proof of Claim 3. By construction,

$$x_1^{(\ell+1)} = x_1^{(\ell)} - (0, e^{(\ell)}) = y_1^{(\ell)} + (0, d^{(\ell)}) = y_1^{(\ell+1)}$$

and by Claim 1 and the special case the result follows. \dashv

Combining the claims yields the sequence of equalities

$$C(x_1, \dots, x_{k+1}) = C(x_1^{(\ell+1)}, \dots, x_{k+1}^{(\ell+1)}) = C(y_1^{(\ell+1)}, \dots, y_{k+1}^{(\ell+1)}) = C(y_1, \dots, y_{k+1})$$

which completes the proof. \square

3.2. Trivialising in stages. The previous proposition shows that C is well-behaved under equal sums of sequences of equal length. In order to fully trivialise C to a Borel coboundary, we require more, both algebraically and combinatorially. Most of the arguments in this subsection are of algebraic type, and hence carry over from \mathbb{R} to \mathbb{R}^2 ; in this subsection, we reproduce lemmas from [22] with additional detail, and follow their general proof structure.

Accordingly, we introduce some notation: given a sequence x_1, \dots, x_n , denote it by $(\bar{x})_n$, and hence let $\sum(\bar{x})_n = x_1 + \dots + x_n$. We also write $(\bar{x})_n \in p^*$ if $x_i \in p^*$ for all $i \leq n$, and $C((\bar{x})_n) = C(x_1, \dots, x_n)$. When the length of $(\bar{x})_n$ is clear, we also simply write \bar{x} in its place. Further, if $k < \omega$ and $g \in G$, write $kg = g + \dots + g$ containing k -many terms.

The following two combinatorial lemmas are due to [22]. As results in this subsection, as can be seen from their proofs, hold for all \mathbb{R}^N ; we state some without proof and refer to [22] for details.

Lemma 3.5 ([22], Lemma 53). *Suppose $(\bar{x})_n, (\bar{y})_m, (\bar{x}')_{n'}, (\bar{y}')_{m'} \in p^*$ where $1 \leq m < n$ and $1 \leq m' < n'$. Suppose $\sum(\bar{x})_n = \sum(\bar{y})_m = s$ and $\sum(\bar{x}')_{n'} = \sum(\bar{y}')_{m'} = s'$. Then*

$$(n' - m')(C(\bar{x}) - C(\bar{y})) = (n - m)(C(\bar{x}') - C(\bar{y}')).$$

Lemma 3.6. *There exist $\tilde{x}, \tilde{y} \in p^*$ and $n < \omega$ such that $(n + 1)\tilde{x} = n\tilde{y}$.*

Proof. Fix any $x \in p^*$ and consider the straight line L containing the origin and x . Since p is open, there exists $n < \omega$ for which $(1 + \frac{1}{n})x \in p^* \cap L$, which yields the desired y . \square

Definition 3.7. *The pair of sequences $(\bar{x})_{(n+1)}$ and $(\bar{y})_n$ given by $x_i = \tilde{x}$ and $y_i = \tilde{y}$ for all i is the **canonical pair**, denoted by (λ, μ) . The **canonical group element** is given by*

$$\tilde{g} = C(\lambda) - C(\mu).$$

Lemma 3.8. *For any $(\bar{x})_n, (\bar{y})_m \in p^*$ with $n \geq m$ and $\sum \bar{x} = \sum \bar{y}$, we have $C(\bar{x}) - C(\bar{y}) = (n - m)\tilde{g}$.*

Proof. Theorem 3.6 implies $(n + 1 - n)(C(\bar{x}) - C(\bar{y})) = (n - m)(C(\lambda) - C(\mu)) = (n - m)\tilde{g}$. \square

Following [22], we define $C_1: (\mathbb{R}^2)^2 \rightarrow G$ given by

$$C_1(x, y) = C(x, y) - \tilde{g}.$$

By induction, we have $C_1((\bar{x})_{(n+1)}) = C((\bar{x})_{(n+1)}) - n\tilde{g}$. Now, observe that $C_1(x, y) - C(x, y) = -\tilde{g}$, a Borel coboundary: to make this explicit, write $B(x) = -\tilde{g}$ so that

$$C_B(x, y) = B(x) + B(y) - B(x + y) = -\tilde{g}.$$

Hence it suffices to prove that C_1 is a Borel coboundary. Luckily, C_1 satisfies stronger closure properties than C .

Lemma 3.9 ([22] Corollary 54). *If $(\bar{x})_n, (\bar{y})_m \in p^*$ satisfy $\sum(\bar{x})_n = \sum(\bar{y})_m$ and $n \geq m$ then $C_1(\bar{x}) = C_1(\bar{y})$.*

Proof. By the inductive remark, theorem 3.8 proves

$$\begin{aligned} C_1(\bar{x}) - C_1(\bar{y}) &= C(\bar{x}) - C(\bar{y}) - (n-1)\tilde{g} + (m-1)\tilde{g} \\ &= C(\tilde{x}) - C(\tilde{y}) - (n-m)\tilde{g} = 0. \end{aligned} \quad \square$$

3.3. Trivialising on a closed set. From the difference in complexity between our theorem 3.3 and Lemma 52 in [22], it is clear that the case \mathbb{R}^2 is not an immediate consequence of the theorem in the case \mathbb{R} . The topological differences between the spaces are emphasised in this section, where we overcome obstacles not present in the 1-dimensional case. These can be reduced to the properties of a set $t(x)$ associated with each $x \in \mathbb{R}^2$. Trivialising on $t(x)$ —similar to trivialising on $[M, \infty)$ in [22]—suffices to prove the theorem.

We begin by identifying a well-behaved subset of \mathbb{R}^2 on which the invariance properties of C_2 apply more broadly. For $k < \omega$, let

$$k \cdot p = \{ (kx_1, kx_2) \mid (x_1, x_2) \in p \}.$$

As we assumed that p lies in the first quadrant, there exist $M < \omega$ and distinct straight lines L_1, L_2 such that:

- there exists a straight line L through the origin for which, for some $\tilde{z} = (z_1, z_2) \in \mathbb{R}^2$, the infinite line segment

$$L^{\tilde{z}} = \{ (x_1, x_2) \in L \mid x_1 \geq z_1 \wedge x_2 \geq z_2 \}$$

is contained in $p^+ = \bigcup_{k > M} k \cdot p$;

- L_1 and L_2 are neither equal nor parallel, $L_1 \cap L_2 \subset p^+$, and the region bounded by L_1, L_2 which is wholly contained in the first quadrant is also contained in p^+ .

We show below that such points and lines always exist. We need some notation. In \mathbb{R}^N , denote the origin by O . Given a condition p , denote its topological closure by \bar{p} , and its topological boundary by ∂p . Let d denote the Euclidean metric.

Lemma 3.10. *Such $M < \omega$, straight lines L, L_1, L_2 , and $\tilde{z} \in \mathbb{R}^2$ exist.*

Proof. Consider the sequence $(k \cdot p)_{k < \omega}$ of conditions (read open sets), where each $k \cdot p$ has vertices $v_k^1, v_k^2, v_k^3, v_k^4$, labelled anti-clockwise starting with the vertex closest to the origin. Since p is open, we may assume that p is not a square. Hence suppose that $\pi_0(v_1^2) - \pi_0(v_1^1) > \pi_1(v_1^3) - \pi_1(v_1^2)$ (i.e. p is wider than it is high); the other case is similar. Define auxiliary straight lines L'_1, L'_2 by

$$\{O, v_1^1\} \subset L'_1 \text{ and } \{O, v_1^3\} \subset L'_2.$$

Note that $\{v_k^1 \mid k < \omega\} \subset L'_1$ and $\{v_k^3 \mid k < \omega\} \subset L'_2$. Let v' be the unique point at which $\partial p \setminus \{v_1^3\}$ and L'_2 intersect. Now, define w to be the unique $v \in \partial p$ for which $d(v_1^1, v) = d(v', v)$ and for which this distance is minimal. That is, $w \in \partial p$ and sits exactly halfway between v_1^1 and v' . Let L be the straight line satisfying

$$\{O, w\} \subset L.$$

To see that for some $M < \omega$ we have $(M \cdot p) \cap ((M+1) \cdot p) \neq \emptyset$, assume w.l.o.g. that p is wider than it is high. Write $v_1^1 = (x_1, x_2)$ and $v_1^4 = (y_1, y_2)$. Then $M < \omega$ is as needed if

$$Mx_1 < (M+1)x_1 < My_1 \iff M > \frac{x_1}{y_1 - x_1}.$$

To complete the proof, fix $\tilde{z} \in L \cap (M \cdot p)$ for a sufficiently large M . Choose L_1 to be the unique line parallel to L'_1 containing \tilde{z} , and define L_2 similarly. \square

In the 2-dimensional case, the definition of T is best given in polar coordinates: let L_1 be the unique straight line through O with angle θ_1 , and define L_2 and θ_2 similarly. Suppose that $\tilde{z} = (s, \varphi)$.

Definition 3.11. *With \tilde{z} , L_1 , L_2 , and L given as above, define*

$$T = \{ (r, \theta) \in \mathbb{R}^2 \mid r \geq s \wedge \theta_2 \leq \theta \leq \theta_1 \}.$$

One can think of T as the *cone* induced by the lines L'_1, L'_2 shifted to \tilde{z} . Observe that $\tilde{z} \in T$. By construction, we may choose \tilde{z} to not be contained in p , which we assume from now on. Hence $T \cap p = \emptyset$.

Lemma 3.12. *With T defined as above, the following hold:*

- (a) *T is non-empty and closed (hence in particular Borel).*
- (b) *$L^{\tilde{z}} \subset T$.*
- (c) *T is closed under addition.*
- (d) *The interior of T has infinite diameter: for every $K < \omega$ there exists $y \in T$ such that the open ball $B_K(y)$ of radius K and centre y is contained in T .*

Proof. These are all routine. It is clear from the construction that (a) T is non-empty and closed (it is the closure of the area bounded by L_1 and L_2 in the direction of (∞, ∞)), and similarly (b) is immediate. The fact (c) that T is closed under addition follows easily from the construction of T and by using polar coordinates. Since L_1 and L_2 are not parallel, the interior of T must have infinite diameter, proving (d). \square

Importantly, every real in T can be expressed as a sum of generics:

Lemma 3.13. *If $z \in T$ then there exists a sequence of generics $(\bar{x})_n$ for which $z = \sum \bar{x}$.*

Proof. Let $z = 2x$ where $x \in p \setminus p^*$. We find $x_1, x_2 \in p^*$ for which $z = x_1 + x_2$. Let $r < p$ contain x , and let ϵ be M -generic and sufficiently small so that $x_1 = x + \epsilon$ and $x_2 = x - \epsilon$ are elements of r^* . Now $z = x_1 + x_2$ as required, and it follows by induction that every $z \in k \cdot p$ (for any $k \geq 2$) can be written in the required form. Since $T \subset p^+$, the lemma is proven. \square

We begin to trivialise cocycles to coboundaries. As in [22], theorem 3.9 shows that the map $F: \mathbb{R}^2 \rightarrow G$ defined by

$$F(x) = \begin{cases} C_1(x_1, \dots, x_n) & \text{if } x = \sum (\bar{x})_n \in T \\ 0 & \text{otherwise} \end{cases}$$

for $(\bar{x})_n \in p^*$ is well-defined. Further, as C_1 is Borel, the graph of F is clearly analytic; hence F is a Borel function and C_F is a Borel coboundary. Now define

$$C_2(x_1, x_2) = C_1(x_1, x_2) + C_F(x_1, x_2).$$

Lemma 3.14. *If $(\bar{x})_n \in p^*$ and $\sum (\bar{x})_n \in T$ then $C_2(\bar{x}) = 0$.*

Proof. By induction. For $n = 2$, it is easily seen that $C_2(x_1, x_2) = F(x_1) + F(x_2) = 0$ by definition of F and since $T \cap p = \emptyset$. Thus, theorem 2.4 implies

$$\begin{aligned} C_2(x_1, \dots, x_{n+1}) &= C_1(x_1, \dots, x_{n+1}) + C_F(x_1, \dots, x_{n+1}) \\ &= C_1(x_1, \dots, x_{n+1}) - F(x_1 + \dots + x_{n+1}). \end{aligned} \quad \square$$

Importantly, the invariance of C_2 on generics in fact extends to all of T , which follows from the combinatorial properties of abelian cocycles.

Corollary 3.15 ([22] Lemma 55). *If $x, y \in T$ then $C_2(x, y) = 0$.*

Proof. Suppose $x = \sum(\bar{x})_n$ and $y = \sum(\bar{y})_m$, all generics in p (this is possible by theorem 3.13). By the algebraic properties of C_2 and the previous lemma we have

$$C_2(\bar{x}, \bar{y}) - C_2(\bar{x}) - C_2(\bar{y}) = C_2(x, y)$$

where all terms on the left-hand side equal 0. \square

We are now ready to define the aforementioned class of families $t(x)$.

Definition 3.16. *For every $x \in \mathbb{R}^2$, let $t(x) \subset \mathbb{R}^2$ be defined by*

$$t(x) = \{ y \in T \mid x + y \in T \}.$$

Lemma 3.17. *Let $x, x', w \in \mathbb{R}^2$.*

- (a) $t(x) \neq \emptyset$
- (b) $t(x) \cap t(x') \neq \emptyset$
- (c) *If $x, x' \in T$ then there exist $z, z' \in t(x) \cap t(x') \cap t(w)$ for which we have $x + z = x' + z'$.*

Proof. For (a), let $x \in \mathbb{R}^2$ be given. If $x \in T$ then $x + x \in T$, thus $x \in t(x)$. Similarly, if $x \in L \setminus T$ (with L defined as in theorem 3.10), take a sufficiently large $y \in L^{\tilde{z}} \subset T$ satisfying $x + y \in T$. Hence suppose $x \in \mathbb{R}^2 \setminus (T \cup L)$. Recall that T has infinite diameter (theorem 3.12 (d)). Hence let $z \in T$ be such that $B_{|x|+1}(z) \subset T$. Then $x + z \in B_{|x|+1}(z) \subset T$. Part (b) follows easily; in fact, the result holds for all finite intersections by a similar argument.

For part (c), let $T_x = \{ x + y \mid y \in T \}$ be the shift of T to x . Since L_1 and L_2 (as defined in theorem 3.10) are not parallel, it is easily seen that if $x, x' \in T$ then $T_x \cap T_{x'} \neq \emptyset$, and in fact also has infinite diameter. As in (a) above, let $v \in T$ be such that $B_{|w|+1}(v) \subset T_x \cap T_{x'}$. Then any element in $B_{|w|+1}(v)$ can be decomposed as desired. \square

Since the set T is closed by construction, the following is immediate.

Corollary 3.18. *For every $x \in \mathbb{R}^2$, the set $t(x)$ is Borel.*

Define $J(x) = C_2(x, \tilde{x})$ where $\tilde{x} \in t(x)$. By our construction, the choice of \tilde{x} is inconsequential, highlighting the C_2 -invariance on T . To see this, first recall that

$$C_2(x, y) = C_1(x, y) - F(x + y) = C(x, y) - \tilde{g} - F(x + y) = C_2(y, x)$$

since \mathbb{R}^2 and G are abelian, and since C is an abelian cocycle.

Lemma 3.19. *J is well-defined and a Borel function.*

Proof. By theorem 2.2,

$$J(x) = C_2(x, \tilde{x}) = C_2(\tilde{x}, x) = C_2(\tilde{x}, z) + C_2(\tilde{x} + z, x) - C_2(\tilde{x} + x, z)$$

for every $z \in \mathbb{R}^2$. Hence if $z \in t(x)$ then theorem 3.12 (c) and theorem 3.15 imply $J(x) = C_2(\tilde{x} + z, x)$. Now suppose $\tilde{x}, \tilde{x}' \in t(x)$, and, using theorem 3.17 (c), let $z, z' \in t(x) \cap t(\tilde{x}) \cap t(\tilde{x}')$ be such that $\tilde{x} + z = \tilde{x}' + z'$. Then

$$C_2(\tilde{x}, x) = C_2(\tilde{x} + z, x) = C_2(\tilde{x}' + z', x) = C_2(\tilde{x}', x).$$

To see that J is Borel as a map, first note that C_2 is Borel by construction. Hence it suffices to show that the set $\{ t(x) \mid x \in X \}$ has a Borel choice (or *uniformisation*) function. To that end, let $P \subset (\mathbb{R}^2)^2$ be the relation

$$(x, y) \in P \iff y \in t(x).$$

Clearly, P is a Borel subset of $(\mathbb{R}^2)^2$ and, fortunately, every section P_x has a canonical witness, which we identify as follows: fix x and travel along L starting at \tilde{z} towards the point at infinity (∞, ∞) . Then the first $y \in L^{\tilde{z}}$ for which a sufficiently large closed ball centred at y is contained in T does the trick (recalling that T is closed shows that this is well-defined).

To formalise this, consider $L^{\tilde{z}}$ and define the function $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\rho(x) = \min_{|y|} \{y \in L^{\tilde{z}} \mid \overline{B}_{|x|+1}(y) \subset T\}$$

where $\overline{B}_r(x)$ denotes the closed ball of radius r and centre x . It follows from the properties of T from theorem 3.12 that $\rho(x)$ is well-defined and as desired. To see that ρ is Borel, recall that a function is Borel if and only if the pre-image of every basic open set is Borel [23, 11.C]. Fix a condition $q \subset \mathbb{R}^2$; we show $\rho^{-1}[q]$ is Borel using theorem 2.1. We may assume that $q \cap L^{\tilde{z}} \neq \emptyset$ since otherwise $\rho^{-1}[q] = \emptyset$. By definition of ρ we know that $\tilde{z} \notin \text{ran}(\rho)$; hence $q \cap L^{\tilde{z}}$ is homeomorphic to the open interval $(0, 1)$. Let

$$b = (\tilde{z} \oplus \tilde{q}) \oplus (\varphi \oplus \theta_1 \oplus \theta_2)$$

where \tilde{q} codes the limit points of $q \cap L^{\tilde{z}}$; we use this as an oracle to apply theorem 2.1. Given $x \in \mathbb{R}^2$, first compute $|x| + 1$. From the limit points $c_1, c_2 \in L^{\tilde{z}}$ of $q \cap L^{\tilde{z}}$ (which are coded into \tilde{q}) compute the largest radii r_1, r_2 such that $\overline{B}_{r_1}(c_1) \subset T$ and $\overline{B}_{r_2}(c_2) \subset T$, which requires finitely-many jumps from b (recall that $|c_1| < |c_2|$). By construction, $r_1 < r_2$, and if $r_1 < |x| + 1 < r_2$ then $x \in \rho^{-1}[q]$, which proves the result by theorem 2.1. \square

To complete the proof of theorem 3.21, it suffices to show that C_2 is a Borel coboundary. For each $x \in \mathbb{R}^2$ let

$$\tilde{x} = \rho(x) \in t(x)$$

which we call the **canonical witness for x** .

Lemma 3.20. *Let $x, y \in \mathbb{R}^2$. If $z \in t(x) \cap t(x + y)$ then $z + x \in t(y)$.*

Proof. By definition, $z \in t(x)$ implies $z + x \in T$. Since $z \in t(x + y)$ we also know that $z + x + y \in T$. Hence $z + x \in t(y)$. \square

Suppose $x, y \in \mathbb{R}^2$ and choose $z \in t(x) \cap t(y) \cap t(x + y)$. Then

$$\begin{aligned} C_2(x, y) &= C_2(x, z) + C_2(x + z, y) - C_2(x + y, z) \\ &= C_2(x, \tilde{x}) + C_2(x + z, y) - C_2(x + y, \widetilde{x + y}) \\ &= J(x) + C_2(x + z, y) - J(x + y). \end{aligned}$$

Finally, notice that $z \in t(x) \cap t(x + y)$ implies $x + z \in t(y)$ by theorem 3.20. Hence

$$C_2(x + z, y) = C_2(y, x + z) = C_2(y, \tilde{y}) = J(y)$$

and so $C_2(x, y) = C_J(x, y)$. Now,

$$C_J(x, y) = C_2(x, y) = C_1(x, y) + C_F(x, y) = C(x, y) + C_B(x, y) + C_F(x, y)$$

and thus $C(x, y) = C_J(x, y) - C_B(x, y) - C_F(x, y)$, a sum of Borel coboundaries. This suffices by theorem 2.5. Hence we conclude:

Theorem 3.21. *For every group G we have $H_{\text{Bor}}^2(\mathbb{R}^2, G) = 0$.*

4. THE CASE $N > 2$

Let $N > 2$ and G be a group. Suppose $C: (\mathbb{R}^N)^2 \rightarrow G$ is an abelian Borel cocycle with Borel code z and let M be a countable transitive model of ZFC containing z and G as elements. As in section 2.2, we assume there exist N -dimensional conditions $p = I_1 \times \dots \times I_N \subset \mathbb{R}^N$ and $q = J_1 \times \dots \times J_N$ satisfying that each $I_i \subset (0, \infty)$ (so p lies in the N -dimensional “first” quadrant, octant, etc.) and so that there exists $\tilde{g} \in G$ for which $C(x) = \tilde{g}$ for every $x \in (p \times q)^*$. Further, we may of course assume $D(p) < D(q)$ (cf. after theorem 2.8). We also stipulate that $|I_i| \neq |I_j|$ for all $i \neq j$, which is needed in theorem 4.2 below.

We retrace the arguments from section 3. In order to prove the theorem along our arguments in the previous section, we need to prove higher dimensional versions of our topological lemmas. We begin with a high-dimensional version of theorem 3.3:

Proposition 4.1. *If $(\bar{x})_n, (\bar{y})_n \in p^*$ and $\sum \bar{x} = \sum \bar{y}$, then $C(\bar{x}) = C(\bar{y})$.*

Proof. As before, the *special case* is combinatorial and hence applies: if there exists $\ell \leq n$ for which $x_\ell = y_\ell$ then the lemma follows immediately. If $n \in \{2, 3\}$, then we are also done: if $n = 2$, the combinatorial argument in [22] applies; if $n = 3$, then the vectors $(y_1 - x_1), (y_2 - x_2), (y_3 - x_3)$ bound a triangle Δ . Since p is open, the same argument as in theorem 3.2 holds, considering Δ in its induced plane inside p . Hence, assume $n > 3$ and that there exists no pair (i, j) for which $x_i = y_j$. Note that the arguments in theorem 3.4 and the recursion in the proof of theorem 3.3 do not depend on the number of dimensions in the ambient space; they can be carried out identically in \mathbb{R}^N for any $N > 2$. \square

Recall that the results in section 3.2 are purely combinatorial, hence apply immediately in the case $N > 2$. The remaining argument reduces to translating section 3.3 to N . In fact, the only obstacle is to correctly define T ; all subsequent arguments are combinatorial.

Recall the case $N = 2$ and denote the set T defined in section 3 by $T(2)$, to highlight its topological dimension. We defined two auxiliary lines L'_1, L'_2 which induce $T(2)$: it is the space bounded by L'_1, L'_2 in the direction of the limit point (∞, ∞) . Note that $T(2)$ is homeomorphic to the closed upper half plane, which we denote by

$$U^2 = \{ (x_1, x_2) \mid x_2 \geq 0 \}.$$

For each $N > 2$, we define a set $T(N)$ homeomorphic to U^N , the N -dimensional closed upper half plane given by

$$U^N = \{ (x_1, \dots, x_N) \mid x_N \geq 0 \}.$$

With the condition $p \subset \mathbb{R}^N$ fixed as before, it is easily seen that there exists $M < \omega$ for which $(M \cdot p) \cap ((M+1) \cdot p) \neq \emptyset$ (cf. theorem 3.10). Indeed, consider p as an N -dimensional hypercube in \mathbb{R}^N . By assumption, all 2^N -many vertices of p are positive. The existence of $M < \omega$ is derived exactly as in the two-dimensional case: the shortest side of p gives a sufficient bound for M just as in the proof of theorem 3.10. Hence, fix such an M , and consider the induced set p^+ .

From now on, we write $X \cong Y$ to mean that X and Y are homeomorphic. The set of straight lines containing the origin O is given by \mathcal{L}_O .

Proposition 4.2. *There exists a subspace $T(N) \subset p^+ \subset \mathbb{R}^N$, homeomorphic to U^N , which shares the properties of $T(2)$ on \mathbb{R}^2 .*

Before we give the proof, we require a technical lemma. Let R_M denote the N -hypercube given by the intersection $(M \cdot p) \cap ((M+1) \cdot p)$, and define R_k for $k \geq M$ similarly. We assume here that R_M is homeomorphic to D^N , the closed disc of N dimensions. To define $T(N)$, consider auxiliary lines $L'_1, L'_2 \in \mathcal{L}_O$ which bound each R_k for $k \geq M$; we make this precise below. The induced subspace will serve as our space $T(N)$.

To define the relevant notions, we need the following lemma first. Denote the set of vertices of R_M by $V(R_M)$, and define $V(k \cdot p)$ similarly.

Lemma 4.3. *Let $k \geq M$. Then $V(R_k) \cap V(k \cdot p)$ and $V(R_k) \cap V((k+1) \cdot p)$ are distinct singletons.*

Proof. By shifting and shrinking, w.l.o.g., express the set $V(M \cdot p)$ as the set of $\{0, 2\}$ -words of length 2^N , and similarly, $V((M+1) \cdot p)$ as the set of $\{1, 3\}$ -words of length 2^N . Now, a vertex $v \in V(M \cdot p)$ is an element of $(M+1) \cdot p$ if and only if each of its coordinates lies in the interval $[1, 3]$. Clearly, only the vertex $(2, 2, \dots, 2)$ satisfies this. By an identical argument, the only vertex $v \in V((M+1) \cdot p)$ in $M \cdot p$ is $(1, 1, \dots, 1)$. \square

For a subset $A \subset \mathbb{R}^N$, let its interior be denoted by A° .

Lemma 4.4. *There exists a line $L' \in \mathcal{L}_O$ such that, for every $k \geq M$ we have $R_k^\circ \cap L' \neq \emptyset$.*

Proof. Let $v_k \in \mathbb{R}^N$ be as provided by theorem 4.3, where

$$v_k \in V(R_k) \cap V((k+1) \cdot p).$$

By definition of $k \cdot p$, there exists $v^- \in V(p)$ for which $v_k = (k+1)v^-$. Note that v_k is the vertex of $V((k+1) \cdot p)$ closest to O . Since the set $\{v_i \mid i < \omega\}$ lies on a straight line containing O , defining $L' \in \mathcal{L}_O$ to be the unique line containing v_1 suffices. \square

As provided by the previous lemma, fix the straight line L that passes through every R_k . Let B_M be the N -dimensional open ball contained in R_M° with centre on L and of maximal radius so that $\partial B_M \cap \partial R_M \neq \emptyset$. Define a sequence $(B_k)_{k \geq M}$ such that:

- (1) $\partial B_M \cap \partial R_M \neq \emptyset$
- (2) if $L' \in \mathcal{L}_O$ and L' is tangent to B_M then L' is tangent to B_k for every $k \geq M$.

Since all sides of R_k grow by a constant as k increases, a straightforward geometrical argument shows that the latter condition can be satisfied. This also shows that

$$\lim_{k \rightarrow \infty} \text{diam}(B_k) = \infty.$$

The following corollary is now immediate.

Corollary 4.5. *The sequence $(B_k)_{k \geq M}$ induces an N -dimensional hypercone \mathcal{H} . In particular, \mathcal{H} has infinite N -dimensional diameter.*

Since all sides of $(k+1) \cdot p$ are larger than their respective sides of B_k , we see that, once \mathcal{H} enters p^+ , it does not escape: the radius of B_k is bounded above by the shortest side length of R_k , which is bounded above by the shortest side length of $k \cdot p$. By construction, all bounds are strict.

Proof of theorem 4.2. With $p \subset \mathbb{R}^N$ fixed, consider the hypercone \mathcal{H} (induced by the N -sphere B_M) as given by lemmas 4.4 and 4.5. Isolate its *positive filled part*, defined by

$$\begin{aligned} x = (x_1, \dots, x_N) \in \mathcal{H}^+ &\iff \\ &(\exists L' \in \mathcal{L}_O)(x \in L' \wedge (\forall m \leq N)(x_m \geq 0) \wedge L' \cap \partial B_M \neq \emptyset). \end{aligned}$$

It is easily seen that

$$\partial\mathcal{H}^+ = \bigcup \{L \in \mathcal{L}_O \mid |L \cap \partial B_M| = 1\}$$

and hence $\partial\mathcal{H}^+ \cong \mathbb{R}^{N-1}$. And since \mathcal{H}^+ is clearly convex, it has trivial homology for every $k > 0$ [7, 10.8]. By taking the stereographic projection we hence see that $\mathcal{H} \cong U^N$. To define $T(N)$, fix some $\tilde{z} \in L \cap p^+$ (as before, we may assume $\tilde{z} \notin p$) and take the shift $\mathcal{H}^+ + \tilde{z}$ defined by

$$\mathcal{H}^+ + \tilde{z} = \{x + \tilde{z} \mid x \in \mathcal{H}^+\}.$$

Since $\mathcal{H}^+ + \tilde{z} \subset p^+$, we define $T(N) = \mathcal{H}^+ + \tilde{z}$. The important properties of $T(2)$ from lemma 3.12 carry over to $T(N)$ since \mathcal{H}^+ is closed, has infinite N -dimensional diameter and $T(N) \cong U^N$. \square

To complete the proof, note that the family $\{t(x) \mid x \in \mathbb{R}^N\}$ can be defined as in theorem 3.16, and shares the properties of theorem 3.17. The map J (as defined on \mathbb{R}^N) is Borel by the same argument as in theorem 3.19. Since all remaining lemmas in section 3.3 are algebraic, we conclude:

Theorem 4.6. $H_{\text{Bor}}^2(\mathbb{R}^N, G) = 0$ for every $N < \omega$ and every group G .

Proof. The case $N = 1$ is [22, Theorem 49]; $N = 2$ is theorem 3.21. The remaining cases were covered in section 4. Hence the theorem is proved. \square

5. OPEN QUESTIONS

In the present work we have focussed on the special cases of Borel definable group extensions of \mathbb{R}^N by a *countable* group G . If G is uncountable Borel (recall theorem 1.1: G is a Borel subset of some standard Borel space, and the group operation is a Borel function), the forcing arguments in section 3 do not apply as Cohen forcing has the c.c.c. [17].

This obstruction is fundamental [22, p. 265]: based on an idea of Greg Hjorth, one can construct a Borel subgroup $G \leq \mathbb{R}$ and a Borel cocycle C such that C is a Borel coboundary in $H_{\text{Bor}}^2(\mathbb{R}, \mathbb{R})$ but not in $H_{\text{Bor}}^2(\mathbb{R}, G)$ —by theorem 1.2 such a G is necessarily uncountable—which shows that theorem 1.2 fails in general for uncountable G . (It should be noted that a similar construction works for Cantor space 2^ω [6, 21].) This leaves the following question open:

Question 5.1. What can be said about the structure of $H_{\text{Bor}}^2(\mathbb{R}^N, G)$ if G is Borel and uncountable and $N \geq 2$?

Contrasting the relationship of theorem 1.2 with our theorems 3.21 and 4.6, one can consider the impact of the properties of G on $H_{\text{Bor}}^2(\mathbb{R}^N, G)$ for varying N and ask:

Question 5.2. For which (group-theoretic) properties P is the statement

$$H_{\text{Bor}}^2(\mathbb{R}, G) = H_{\text{Bor}}^2(\mathbb{R}^N, G) \text{ whenever } G \text{ has property } P$$

true for all $N < \omega$? For which properties P do both $H_{\text{Bor}}^2(\mathbb{R}, G)$ and $H_{\text{Bor}}^2(\mathbb{R}^N, G)$ vanish for all $N < \omega$ whenever G has property P ?

For instance, in this paper we have shown that the property of “being countable” is sufficient to answer the second part of theorem 5.2 affirmatively.

While Hjorth and Kanovei-Reeken have proven that $H_{\text{Bor}}^2(2^\omega, 2^\omega)$ and $H_{\text{Bor}}^2(2^\omega, G)$ are non-trivial—the latter even for finite G [22, p. 265]—the following questions remain open to the author’s knowledge.

Question 5.3. What can be said about the structure of $H_{\text{Bor}}^2(\omega^\omega, G)$ for countable (or Borel uncountable) G ? What about $H_{\text{Bor}}^2(\omega^\omega, \omega^\omega)$?

Further, one can consider the study of definable group extensions of groups with more “homological” structure and solve the associated classification problem in the context of Borel equivalence relations, orbits, and group actions (see [1, 13] for background). Particular cases in this context have been studied in-depth, and various classification results have been obtained (e.g. the preprint [26] for certain countable groups; in spirit, these are related to theorem 5.2).

However, for other spaces answers are scarce. Take for instance the p -adic integers \mathbb{Z}_p for p prime. This ring can be expressed as the inverse limit of the sequence $(\mathbb{Z}/p, \mathbb{Z}/p^2, \dots)$, and is hence Polish [14]. While progress has been made on related definability problems in terms of homological algebra and Borel equivalence relations (e.g. [2, 3]), one can wonder about the effect of the inverse limit construction of \mathbb{Z}_p on trivialising Borel cocycles, and whether these can be turned into Borel coboundaries in a step-by-step argument along the limit. Further, \mathbb{Z}_p also carries algebraic structure (it is a valuation ring, a local ring, etc.), which could be investigated in this context. So, we ask:

Question 5.4. What can be said about the structure of $H_{\text{Bor}}^2(\mathbb{Z}_p, G)$ for G Borel and countable, or otherwise?

6. ACKNOWLEDGEMENTS

The author’s research was funded by the Singapore Ministry of Education through its research grant with number MOE-000538-01.

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