CO-ANALYTIC COUNTEREXAMPLES TO MARSTRAND'S PROJECTION THEOREM

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Abstract. We show that ZFC does not prove Marstrand's Projection Theorem in $\mathbb{R}^2$ for projective pointclasses beyond the analytic sets. This answer a decades-old open question from classical geometric measure theory. Assuming $V = L$, we construct co-analytic subsets of $\mathbb{R}^2$ which fail Marstrand's Projection Theorem as badly as possible (the difference between the Hausdorff dimension of the set and of that of its orthogonal projections is maximal). We then construct counterexamples of any non-trivial Hausdorff dimension. We use modern tools from the theory of algorithmic randomness and descriptive set theory, including the Point-to-Set Principle and a recursion theorem for co-analytic sets.

1. Introduction

A central goal of descriptive set theory is the characterisation of limits of provability in ZFC. This is done via **definable counterexamples**: sets minimal in the projective hierarchy which fail some property P, while every set of smaller complexity satisfies P. The significance follows from Gödel's Completeness Theorem: if P can consistently fail for a set in pointclass Γ, then ZFC cannot prove P for all sets in Γ. Hence, the least pointclass containing a definable counterexample characterises the limit of provability of P in ZFC.

Since descriptive set theory is influenced by both classical mathematics and logic, the construction of definable counterexamples often requires sophisticated ideas, usually from computability theory and set theory. For example, that ZFC proves the Perfect Set Property for $\boldsymbol{\Sigma}^1_1$ but not for $\boldsymbol{\Pi}^1_1$ sets was shown by Gödel [17, 18] in his groundbreaking work on L, 31 years after Bernstein [3] showed that *some* set fails the Perfect Set Property. This difficulty highlights the strengths of the logical approach: since virtually all mathematics can be formalised within ZFC, all classical theorems can be recast in the language of logic. In subjects of sets of reals (e.g. topology, analysis, geometric measure theory) descriptive set theory then *proves* what can—and cannot—be proven.

In this work, we identify the limit of provability for MP, the **Marstrand Property**, which is related to J. M. Marstrand's landmark projection theorem for Hausdorff dimension. In 1954, Marstrand [38] proved MP for all Borel sets; in 1975, P. Mattila [40] extended the theorem to all $\boldsymbol{\Sigma}^1_1$ sets.

Does ZFC prove MP for all $\boldsymbol{\Pi}^1_1$ sets? We prove this is not so:

Theorem A. ZFC *does not prove* MP *for all* $\boldsymbol{\Pi}^1_1$ *subsets of* $\mathbb{R}^2$.

Therefore, MP is optimal for $\boldsymbol{\Sigma}^1_1$ sets.

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In our proof, we assume $V{=}L$ and build $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets failing MP by recursion. In particular, we show that the Hausdorff dimension of definable counterexamples for MP can be arbitrary.

Theorem B ($V{=}L$)**.** *For every $\epsilon \in [0,1]$ there exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set $E_\epsilon \subset \mathbb{R}^2$ such that we have $\dim_H(E_\epsilon) = 1+\epsilon$ while $\dim_H(\mathrm{proj}_\theta(E_\epsilon)) = \epsilon$ for every $\theta \in [0,2\pi)$.*

1.1. **Background. Marstrand's Projection Theorem** relates the Hausdorff dimension of $E \subseteq \mathbb{R}^2$, written $\dim_H(E)$, to the Hausdorff dimension of its orthogonal projection onto the straight line of angle θ, written $\dim_H(\mathrm{proj}_\theta(E))$. This can be expressed in terms of the regularity property MP (below, "almost all" means "for a set of Lebesgue measure 1"):

Definition 1.1 (MP)**.** A set $E \subseteq \mathbb{R}^2$ has the **Marstrand Property** MP if for almost all $\theta \in [0,2\pi)$:

(1) If $\dim_H(E) \geq 1$ then $\dim_H(\mathrm{proj}_\theta(E)) = 1$.
(2) If $\dim_H(E) < 1$ then $\dim_H(\mathrm{proj}_\theta(E)) = \dim_H(E)$.

In 1954, Marstrand [38] proved MP for all Borel subsets of $\mathbb{R}^2$, assuming ZFC. Generalisations due to P. Mattila [40], who extended the result to $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets (and to higher dimensions), as well as simplifications to the proof due to R. Kaufman [24] (using energy potential characterisations) followed. The combined result of Marstrand and Mattila of MP for $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets is nowadays known as **Marstrand's Projection Theorem**.

Theorem 1.2 (Marstrand's Projection Theorem)**.** *If $E \subseteq \mathbb{R}^2$ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ then E satisfies MP.*

A landmark result of geometric measure theory, refinements of Marstrand's Projection Theorem—and projection theorems in general—remain an important cornerstone of contemporary fractal geometry [2, 15, 14].

Alongside the theorems of Marstrand, Mattila, and Kaufman, constructing sets which *failed* MP yielded eye-catching results, too. In 1979, R. O. Davies [10] constructed a set failing MP using CH. A proof analysis reveals that this set is $\underset{\sim}{\mathbf{\Sigma}}^1_3$.

However, whether Marstrand's Projection Theorem holds for pointclasses *beyond* $\underset{\sim}{\mathbf{\Sigma}}^1_1$ remained open. Partial results proving sufficient conditions for MP to hold have been isolated: the Hausdorff and packing dimension of E being equal is one such condition which is independent of being $\underset{\sim}{\mathbf{\Sigma}}^1_1$ [35]. Further, using a theorem of Crone, Fishman, and Jackson [9], D. Stull [57] also showed that *every* set of reals satisfies MP in $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$. Since determinacy for $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets is not provable in ZFC due to theorems of D. Martin and L. Harrington [23, Theorems 31.4 and 31.5], the following question has remained open:

Is it consistent with ZFC that there exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set which fails MP?

We answer this question in the present paper.

Theorem A. *ZFC does not prove MP for all $\underset{\sim}{\mathbf{\Pi}}^1_1$ subsets of $\mathbb{R}^2$.*

As a corollary we obtain *sharpness*:

Corollary 4.5. *Assuming ZFC, the regularity property MP is sharp for $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subsets of $\mathbb{R}^2$: ZFC proves MP for all $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets, but not for all $\underset{\sim}{\mathbf{\Pi}}^1_1$ subsets.*

Taking ZFC as the basis for reasoning about classical mathematics, Corollary 4.5 shows that Marstrand's Projection Theorem as stated in Theorem 1.2—and hence its

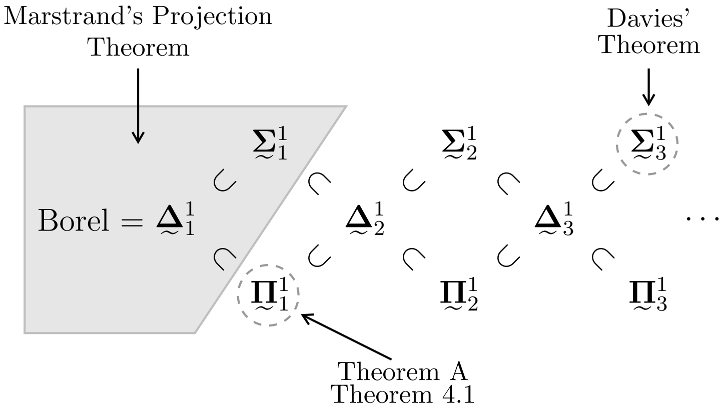

FIGURE 1. The shaded area indicates those pointclasses of $\mathbb{R}^2$ for which ZFC proves MP. The circled pointclasses contain subsets of $\mathbb{R}^2$ failing MP, consistently relative to ZFC.

classical proofs due to Marstrand and Mattila—are optimal: the assumption of $\boldsymbol{\Sigma}^1_1$-ness cannot be strengthened. In ZFC in particular, there is no proof—classical or otherwise—of MP for $\boldsymbol{\Pi}^1_1$ sets.

Our contribution in terms of provability of MP is highlighted in fig. 1, from which optimality is evident (if ZFC is consistent). To obtain Theorem A, we assume the set-theoretical Axiom of Constructibility $V=L$ and explicitly construct a $\boldsymbol{\Pi}^1_1$ set which fails MP. Since Theorem 1.2 proves MP for all $\boldsymbol{\Sigma}^1_1$ sets, this is optimal. This construction is the content of Theorem 4.1:

Theorem 4.1 ($V=L$)**.** *There exists a* $\boldsymbol{\Pi}^1_1$ *set* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *while* $\dim_H(\text{proj}_\theta(E)) = 0$ *for all* $\theta \in [0, 2\pi)$.

We then go one step further: we extend Theorem 4.1 and show how to construct for any $\epsilon \in [0, 1]$ a $\boldsymbol{\Pi}^1_1$ set E such that $\dim_H(E) = 1 + \epsilon$ and E maximally fails MP. This is the content of Theorem B:

Theorem B ($V=L$)**.** *For every* $\epsilon \in [0, 1]$ *there exists a* $\boldsymbol{\Pi}^1_1$ *set* $E_\epsilon \subset \mathbb{R}^2$ *such that we have* $\dim_H(E_\epsilon) = 1 + \epsilon$ *while* $\dim_H(\text{proj}_\theta(E_\epsilon)) = \epsilon$ *for every* $\theta \in [0, 2\pi)$.

This *maximal* failure is explained as follows. Theorem B is optimal not only globally (i.e. with respect to the complexity in the projective hierarchy) but also locally in the following sense: in our proof, we exhibit sets which fail MP as badly as possible. For every $\epsilon \in (0, 1)$ we construct a $\boldsymbol{\Pi}^1_1$ set $E_\epsilon \subset \mathbb{R}^2$ such that $\dim_H(E_\epsilon) = 1 + \epsilon$ while *for every* θ we have $\dim_H(\text{proj}_\theta(E_\epsilon)) = \epsilon$. Hence, our set E_ϵ behaves as pathologically as possible under orthogonal projections; not only does it fail the measure-quantifier "for almost all" in the statement of Definition 1.1, it satisfies the universal negation instead. By classical arguments from geometric measure theory, the drop in Hausdorff dimension from $1 + \epsilon$ to ϵ is maximal.

The proofs of Marstrand, Mattila, and Kaufman are all *classical*: they solely use tools from analysis and geometric measure theory. In contrast, we use modern tools of computability theory to prove Theorem B. For each $\epsilon \in (0, 1)$, we construct a set of reals E_ϵ by recursion, using techniques from algorithmic randomness. In particular, we employ two key tools. First, we use the point-wise characterisation of Hausdorff dimension in terms of **Kolmogorov complexity**, given in its full form by J. Lutz' and

N. Lutz' Point-to-Set Principle [33]. Second, we apply a recursion theorem isolated by Z. Vidnyánszky [58] which guarantees that the set we construct recursively is $\utilde{\mathbf{\Pi}}^1_1$, assuming $V{=}L$. (In particular, without the $\utilde{\mathbf{\Pi}}^1_1$ requirement, J. Miller observed that our argument can be carried out assuming only CH to give a new proof of Davies' theorem.)

1.2. **Relation to Other Work.** Theorems A and B were conceived during the author's PhD thesis work [46]. The results were announced in November 2022, and a preprint was made available shortly after [45]. Concurrently and independently, Theorem B has also been announced by T. Slaman and D. Stull in June 2022 as "in progress" [52]. Unfortunately, their proof has not yet appeared in writing. Their argument seems to use properties of the lightface Π^1_1 sets alongside the computability-theoretic properties of the self-constructible reals (due to Kechris, Guaspari, and Sacks [25, 49]) and a recursion using the Point-to-Set Principle.

Relatedly, also assuming $V{=}L$, Slaman [51] showed that the set of self-constructible reals $\mathcal{C}_1$ has full Hausdorff dimension. Since $\mathcal{C}_1$ contains no perfect subset, his argument yields a Π^1_1 set $E \subset 2^\omega$ which does not contain uncountable closed subsets.

1.3. **The Structure of This Paper.** In Section 2, we introduce geometric measure theory, (descriptive) set theory, and computability and randomness as needed for our arguments.

In Section 3, we prove technical lemmas and outline the techniques which we later employ in the proofs of our main theorems. In Sections 3.1 and 3.2, we prove lemmas relating the Hausdorff dimension of a set of reals given in Euclidean coordinates to the Hausdorff dimension of its representation in polar coordinates. In Section 3.3, we outline Vidnyánszky's $\utilde{\mathbf{\Pi}}^1_1$-Recursion Theorem and how we use it in our constructions.

In Section 4, we prove Theorem 4.1 and our main Theorem A.

In Section 5, we give the proof of Theorem B, which produces $\utilde{\mathbf{\Pi}}^1_1$ counterexamples of MP of arbitrary Hausdorff dimension between 1 and 2.

We close with section 6 where we give a few open questions which illustrate possible avenues for follow-up research.

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2. Preliminaries

In this section, we collect necessary preliminaries on Hausdorff measure and dimension, descriptive set theory, and computability theory and algorithmic randomness.

2.1. **Notation and Conventions.** A **real** is either an element of Cantor space 2^ω—the space of all infinite binary sequences—or of $\mathbb{R}$. The space of **finite binary sequences** is denoted by $2^{<\omega}$. If $\sigma \in 2^{<\omega}$, let $\ell(\sigma)$ denote the **length of** σ. If $\sigma, \tau \in 2^{<\omega}$ and $\ell(\tau) < \ell(\sigma)$ and $\tau(k) = \sigma(k)$ for all $k \in \operatorname{dom}(\tau)$, then τ **is a prefix of** σ, written as $\tau \prec \sigma$. If $\sigma \in 2^{<\omega}$ and $k < \omega$ then $\sigma^k = \sigma \ldots \sigma$, repeated k times. If $f \in 2^\omega$ and $n < \omega$ then $f \upharpoonright n = f(0) \ldots f(n-1)$. The **natural numbers** are denoted by $\omega = \{0, 1, 2, \ldots\}$. If $f \colon \omega \to \omega$ is a function, then its **domain** is denoted by $\operatorname{dom}(f)$.

Strict inclusion is denoted by $\subset$, while inclusion-or-equality is denoted by $\subseteq$. All log are $\log_2$.

2.2. **Hausdorff Measure and Dimension.** Take a subset $E \subseteq \mathbb{R}^2$, and let d denote the usual Euclidean metric. To define Hausdorff measure, we first consider the s-dimensional Hausdorff outer measure of weight δ, given by

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{i<\omega} |U_i|^s \;\middle|\; E \subseteq \bigcup_{i<\omega} U_i \wedge (\forall i < \omega)(|U_i| < \delta) \right\}$$

where $|U| = \sup\{d(x,y) \mid x, y \in U\}$, the **diameter of** U. As δ increases, we include more covers in the infimum, and hence if $0 < \delta < \delta'$ then $\mathcal{H}^s_{\delta'} < \mathcal{H}^s_\delta$. So, as δ decreases, the term $\mathcal{H}^s_\delta(E)$ increases. In particular, the limit $\lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$ always exists, though it might be infinite.

Let $E \subseteq \mathbb{R}^2$. The **s-dimensional Hausdorff outer measure of** E is

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E).$$

The outer measure $\mathcal{H}^s$ is in fact a metric outer measure, and it is defined on all subsets of $\mathbb{R}^2$ [41, p. 55]. We note here that for integer s and sufficiently regular E, we have that $\mathcal{H}^s(E)$ equals the s-dimensional Lebesgue measure, up to a constant [41, p. 58]. Further, one can show that for every $E \subseteq \mathbb{R}^2$ there exists a critical value for s at which $\mathcal{H}^s(E)$ drops from ∞ to 0—this is the **Hausdorff dimension of** E:

$$\begin{aligned}\dim_H(E) &= \sup\{s \geq 0 \mid \mathcal{H}^s(E) = \infty\} \\ &= \inf\{s \geq 0 \mid \mathcal{H}^s(E) = 0\}.\end{aligned}$$

Although, by the Hahn-Kolmogorov-Theorem [11, p. 40], $\mathcal{H}^s$ is a measure only on the class of measurable sets, every subset of $\mathbb{R}^2$ has a Hausdorff dimension [41, pp. 58–59].

A map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is **Lipschitz with constant** $M > 0$ if

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}^m$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$. Importantly for our arguments later on, Lipschitz maps do not increase Hausdorff dimension.

Lemma 2.1. *Let $E \subseteq \mathbb{R}^2$. If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is Lipschitz then*

$$\dim_H(f(E)) \leq \dim_H(E).$$

As a special case, isometries fix Hausdorff dimension:

Corollary 2.2. *Hausdorff dimension is preserved under rotation and translation.*

2.3. **(Descriptive) Set Theory.** We use the hierarchies of descriptive set theory to measure limits of provability.

Consider the Polish (i.e. separable and completely metrisable) space $\mathbb{R}$ endowed with its usual topology. Since $\mathbb{R}$ is metrisable and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a countable basis of open sets $\{U_n \,|\, n < \omega\}$ for the topology of $\mathbb{R}$. A set $E \subseteq \mathbb{R}$ is **open** if there exists $A \subseteq \omega$ such that $E = \bigcup\{n \in A \,|\, U_n\}$. The complement of an open set is **closed**. A set is **Borel** if it is contained in the σ-algebra generated by the open sets. By a classical theorem of Souslin [26, II.14.2], the Borel sets are not closed under projections. A set A is $\utilde{\mathbf{\Sigma}}^1_1$, or **analytic**, if there exists a closed set $C \subset \mathbb{R} \times \omega^\omega$ such that $x \in A$ if and only if there exists $f \in \omega^\omega$ for which $(x, f) \in C$. The complement of a $\utilde{\mathbf{\Sigma}}^1_1$ set is called $\utilde{\mathbf{\Pi}}^1_1$, or **co-analytic**. Sets obtained by taking projections of $\utilde{\mathbf{\Pi}}^1_n$ sets are called $\utilde{\mathbf{\Sigma}}^1_{n+1}$; hence, every $\utilde{\mathbf{\Sigma}}^1_n$ set is $\utilde{\mathbf{\Sigma}}^1_{n+1}$. If a set is both $\utilde{\mathbf{\Sigma}}^1_n$ and $\utilde{\mathbf{\Pi}}^1_n$ then it is $\underset{\sim}{\mathbf{\Delta}}^1_n$. A set is called **projective** if it is $\underset{\sim}{\mathbf{\Sigma}}^1_n$ for some $n < \omega$. By constructing universal sets, one sees that this **projective hierarchy** does not collapse: there exists a $\underset{\sim}{\mathbf{\Sigma}}^1_{n+1}$ set which is not $\underset{\sim}{\mathbf{\Sigma}}^1_n$.

We will need the **Axiom of Constructibility**, denoted by $V{=}L$. The constructible hierarchy is defined by recursion on the ordinals:

- $L_0 = \emptyset$
- $L_{\alpha+1} = \{x \subseteq L_\alpha \mid x \text{ is definable over } (L_\alpha, \in) \text{ with parameters}\}$
- $L_\lambda = \bigcup\{L_\beta \mid \beta < \lambda\}$ if λ is a limit ordinal.

The axiom $V{=}L$ now says that every set is constructible:

$$(\forall x)(\exists \alpha)(x \in L_\alpha)$$

It is known due to Gödel that $V{=}L$ proves both the Axiom of Choice AC and the Continuum Hypothesis CH (and even the Generalised Continuum Hypothesis) [22, Theorems 13.18 and 13.20].

For further details on descriptive set theory, we refer the reader to the textbooks of Y. Moschovakis and A. Kechris [44, 26]. For classical set theory, we recommend Jech [22].

2.4. **Computability and Algorithmic Randomness.** We consider the theory of computability on ω. A function $f\colon \omega \to \omega$ is **partial computable** (p.c.) if there exists a Turing machine M for which, if M on input k halts in finite time, then it outputs the value $f(k)$. The subset of ω on which M halts is the **domain of** f. If the domain of f is ω, then we call f **computable**. A set $A \subseteq \omega$ is m**-reducible** to $B \subseteq \omega$ if there exists a computable function f such that $x \in A$ if and only if $f(x) \in B$. Via classical codings such as Gödel coding we can define computable functions on $2^{<\omega}$.

We require the notion of information density, which we describe in terms of **prefix-free Kolmogorov complexity**. A function $f\colon 2^{<\omega} \to 2^{<\omega}$ is **prefix-free** if whenever $\sigma \in \mathrm{dom}(f)$ and $\tau \prec \sigma$ then $\tau \notin \mathrm{dom}(f)$. Let U be a universal prefix-free machine [12, 3.5]. The machine U satisfies the following: every prefix-free p.c. function $f\colon 2^{<\omega} \to 2^{<\omega}$ has a program code p for which $U(p, x) = f(x)$. The requirement of prefix-freeness does not impose any restrictions since every p.c. function can be made prefix-free [12, Proposition 3.5.1]. If $A \in 2^\omega$, let U^A be the universal prefix-free machine that has access to the oracle A (i.e., the machine can perform a step of the type "does k belong to A?" for any $k < \omega$, and branch accordingly). If $\tau \in 2^{<\omega}$, let U^τ denote the universal prefix-free machine with τ written on an additional tape, the **condition tape**.

Formalising the above we obtain:

Lemma 2.3. *Let $A \in 2^\omega$. There exists a function $h_A\colon \omega \to \omega$ such that, if p is a prefix-free program for the prefix-free A-p.c. function f then*

$$h_A\left(0^{\ell(p)}1px\right) = U^A(p, x) = f(x).$$

If $\tau \in 2^{<\omega}$, define h_τ similarly, using the condition tape in place of the oracle.

We now define formal notions of complexity and randomness on finite strings. These are independently due to A. N. Kolmogorov and R. J. Solomonoff [27, 55].

Definition 2.4. Let $\sigma \in 2^{<\omega}$. The **Kolmogorov complexity of** σ is

$$K(\sigma) = \min\{\ell(\rho) \mid h_\emptyset(\rho) = \sigma\}.$$

If $\tau \in 2^{<\omega}$ then the **conditional Kolmogorov complexity** of σ given τ is

$$K(\sigma \mid \tau) = \min\{\ell(\rho) \mid h_\tau(\rho) = \sigma\}.$$

If A is an oracle, $K^A(\sigma)$ is defined analogously, with h_A in place of $h_\emptyset$. In that case, the condition tape is empty (accessing the oracle does not require any tapes), and hence we can define $K^A(\sigma \mid \tau)$ as above.

Prefix-freeness shows K **is subadditive up to a constant**: for $\sigma, \tau \in 2^{<\omega}$,

$$K(\sigma\tau) \leq K(\sigma) + K(\tau) + c.$$

Similarly, by giving a prefix-free representation of σ, some constant c:

$$K(\sigma) \leq \ell(\sigma) + 2\log(\ell(\sigma)) + c.$$

The notion of Komogorov complexity can be carried over to infinite binary sequences.

Definition 2.5. Let $f \in 2^\omega$ and $A \subseteq \omega$. Then f is **Kolmogorov random relative to** A (or A-random) if there is a constant c such that for all $n < \omega$,

$$K^A(f \upharpoonright n) \geq n - c.$$

As a result of prefix-freeness of U, A-randomness coincides with a number of other randomness notions, highlighting its "correctness" [39, 50, 28, 7]. Via the connection to Martin-Löf randomness, for example, one sees that the set of A-random reals has full Lebesgue measure as a subset of 2^ω. Hence:

Lemma 2.6. *Let $A \in 2^\omega$ and $\sigma \in 2^{<\omega}$. There is $f \in 2^\omega$ such that $\sigma \prec f$ and f is A-random. In particular, for every $A \in 2^\omega$ there exists an A-random real.*

As standard references for Kolmogorov complexity, we refer the reader to the books of R. Downey and D. Hirschfeldt [12] and M. Li and P. Vitány [29]. For more on random reals, see in particular [12, 6.2]. For classical computability, see the texts of R. Soare [54, 53].

2.5. **Coding Objects in** $2^{<\omega}$**.** We identify certain elements of $2^{<\omega}$ with objects in the domain of discourse; these are usually elements of $\mathbb{Q}$ and elements of ω. This identification takes place in the meta-theory; however, determining whether $\sigma \in 2^{<\omega}$ codes a rational or natural number is computable. We denote this string representation using an overline: if x is an object in the domain of discourse, then $\overline{x} \in 2^{<\omega}$ denotes its coded representation.

We fix a particular coding below, with concatenation the implied operation.

- If $k < \omega$ then let $\overline{k}$ be the string whose digits are the binary expansion of k.
- If $n \in \mathbb{Z}$ then let w be the binary expansion of n with each digit doubled (e.g. $n = 101$ becomes $w = 110011$). Then let $\overline{n} = w01$ if $n \geq 0$, and otherwise let $\overline{n} = w10$.
- If $q \in \mathbb{Q}$ then suppose $q = a/b$. Then let $\overline{q} = \overline{a}\overline{b}$.
- If $q = (q_1, \ldots, q_m) \in \mathbb{Q}^m$ then let $\overline{q} = \overline{q_1} \cdots \overline{q_m}$.
- If $x \in \mathbb{R}$, suppose $k < \omega$, and express x in binary. Take the integer part of x and double each digit; denote this string by w. Take the first k bits of x after the binary point, denoted by z. If $x \geq 0$, let $\overline{x}[k] = w01z$; otherwise define $\overline{x}[k] = w10z$.
- If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, suppose $k < \omega$. Then $\overline{x}[k] = \overline{x_1}[k] \cdots \overline{x_m}[k]$.
- If $x \in \mathbb{R}$ then let $\overline{x} \in 2^\omega$ be the limit of $\overline{x}[k]$ in the obvious fashion. If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ then interweave $\overline{x_1}, \ldots, \overline{x_m}$ bit by bit.

Using this coding, if $k < \omega$ then $\ell(\overline{k}) \leq \log(k) + 1$.

The distinction between strings and objects matters when we discuss elements of $\mathbb{R}$ and their truncated approximations. In other cases we are more casual; e.g. we write $K(k)$ and $K(q)$ instead of the formally correct $K(\overline{k})$ and $K(\overline{q})$.

2.6. **Dimension of Points and the Point-to-Set Principle.** Using effective tools to answer measure-theoretical questions has been a research avenue for decades. This development goes back at least to the beginning of this century, with contributions by J. Lutz and Mayordomo [30, 42, 32]. Further discoveries were made by Hitchcock [19, 20] and Mayordomo [42], who related the notion of effective dimension of reals (in 2^ω) to Kolmogorov complexity (the connection between martingales and Hausdorff dimension had previously been investigated by Ryabko [47, 48], Staiger [56], and Cai and Hartmanis [5]; gales, a generalisation of martingales, are due to Lutz [31]).

Definition 2.7. Let $f \in 2^\omega$ and $A \subseteq \omega$. The **(effective) dimension of f relative to A** is given by

$$\dim^A(f) = \liminf_{n\to\infty} \frac{K^A(f \upharpoonright n)}{n}$$

This notion can be naturally extended to m-dimensional Euclidean space $\mathbb{R}^m$: first, consider the complexity of a point.

Definition 2.8. Let $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $A \subseteq \omega$. The **Kolmogorov complexity of x at precision $t < \omega$ relative to A** is given by

$$K_t^A(x) = \min\{K^A(q) \mid q \in \mathbb{Q}^m \cap B_{2^{-t}}(x)\}$$

where $B_s(y)$ is the open ball with respect to the Euclidean metric, with radius s and centre y. The **effective Hausdorff dimension of x relative to A** is

$$\dim^A(x) = \liminf_{t\to\infty} \frac{K_t^A(x)}{t}.$$

The characterisation of effective Hausdorff dimension of reals given in Theorem 2.8 is due to Mayordomo [42]. We can now state Lutz' and Lutz' Point-to-Set Principle [33, Thm. 1].

Theorem 2.9 (Point-to-Set Principle)**.** *Let $n < \omega$ and suppose $E \subseteq \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A\in 2^\omega} \sup_{x\in E} \dim^A(x).$$

The Point-to-Set Principle allows to control the dimension of a set by focussing on individual points. Lutz and Lutz [33] and Lutz and Stull [35] provide outlines and applications of the Point-to-Set Principle. Recently, the Point-to-Set Principle has been extended to arbitrary separable metric spaces [34].

Crucial to our arguments is the following technical lemma, which allows us to work with finite strings instead of approximating rationals. As we will be working with elements in Cantor space 2^ω whilst talking about reals in $\mathbb{R}$, a convenient identification is useful. We use the following observation by Lutz and Stull [35, p. 6].

Lemma 2.10 ([36, Corollary 2.4])**.** *For every $m < \omega$ there exists a constant c such that for all $t < \omega$ and $x \in \mathbb{R}^m$ we have*

$$|K_t(x) - K(\overline{x}[t])| \leq K(t) + c.$$

We note an important corollary.

Corollary 2.11. *If $m \geq 1$ and $x \in \mathbb{R}^m$ then* $\dim(x) = \liminf_{r\to\infty} \frac{K(\overline{x}[r])}{r}$.

In Proposition 3.1, we prove an identification argument explicitly for polar coordinates.

3. Technical Lemmas in Our Construction

3.1. **Arguing in Polar Coordinates.** In our constructions in both Theorems B and 4.1, we work in polar coordinates instead of Euclidean coordinates. A point $(x, y) \in \mathbb{R}^2$ **has polar coordinates** (r, θ) if we have $x = r\cos\theta$ and $y = r\sin\theta$. We restrict our attention to the first quadrant of the unit disc, which we denote by

$$\mathbb{D} = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y \geq 0 \wedge \sqrt{x^2 + y^2} \leq 1\right\}.$$

Thus, $r \in [0, 1]$ and $\theta \in [0, \pi/2]$. Importantly, all points expressed in the proofs below are given in Euclidean coordinates. When we write (r, θ) we do *not* mean the point $(r\cos\theta, r\sin\theta)$.

The following lemma can be considered a direct analogue to Lemma 2.10; its proof is related to that Corollary 2.4 from [36], however requires further tools.

Proposition 3.1. *Let $A \in 2^\omega$. Suppose $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ). Then*

$$\dim^A(x, y) = \dim^A(r, \theta).$$

We provide proofs to both directions of Proposition 3.1 below. We use the following lemma due to Casey and J. Lutz [6].

Lemma 3.2. *There exists $c < \omega$ such that for all $m, s, \Delta s < \omega$ and all $x \in \mathbb{R}^m$:*

$$K_s(x) \leq K_{s+\Delta s}(x) \leq K_s(x) + K(s) + c_m(\Delta s) + c$$

where

$$c_m(\Delta s) = K(\Delta s) + m\Delta s + 2\log\left(\left\lceil \frac{1}{2}\log(m)\right\rceil + \Delta s + 3\right) + \left(\left\lceil \frac{1}{2}\log(m)\right\rceil + 3\right)m + K(m) + 2\log(m).$$

Observe that the term $c_m(\Delta s)$ does not depend on s.

Before give the proof of Proposition 3.1, we require the following standard result [4, p. 151].

Lemma 3.3. *Let $C \subset \mathbb{R}^2$ be convex and compact. If $f : C \to \mathbb{R}^2$ sending (x, y) to $f(x, y)$ is continuously differentiable on C then it is Lipschitz on C.*

The first halves of the proofs below follow the argument of Lutz and Stull [36, Lemma 2.3]. Note that the map $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ is continuously differentiable everywhere, and that $[0, 1] \times [0, \pi/2]$ is compact and convex.

Lemma 3.4 (First half of Proposition 3.1)**.** *There exists c such that if $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) then for all s:*

$$K_s(x, y) \leq K(\overline{r}[s]\overline{\theta}[s]) + K(s) + c.$$

Proof. By Lemma 3.3, the map converting polar into Cartesian coordinates is Lipschitz on $[0,1] \times [0,\pi/2]$. Thus, there exists $M > 0$ such that whenever $(r,\theta),(r',\theta') \in [0,1] \times [0,\pi/2]$ then

$$|(r\cos\theta, r\sin\theta) - (r'\cos\theta', r'\sin\theta')| \leq M|(r,\theta) - (r',\theta')|. \tag{3.1}$$

Let $(r,\theta) \in [0,1] \times [0,\pi/2]$, and suppose that $(x,y) = (r\cos\theta, r\sin\theta)$. Let $r_s = r[s]$, and $\theta_s = \theta[s]$, the truncations of r and θ to s bits after the binary point. We consider the approximation $r_s \cos\theta_s$ of $r\cos\theta$, and similarly $r_s \sin\theta_s$ of $r\sin\theta$. Since these need not be finite strings, we define

$$x[s] = (r_s \cos\theta_s)[s] \qquad \text{and} \qquad y[s] = (r_s \sin\theta_s)[s]$$

which are the truncations to s bits after the binary point. We now approximate (x,y):

Claim 1. $(x[s], y[s]) \in B_{2^{-s}(1+M\sqrt{2})}(x,y)$

Proof of Claim 1. Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Hence,

$$|(x[s], y[s]) - (r_s\cos\theta_s, r_s\sin\theta_s)| \leq 2^{-s}.$$

Using eq. (3.1) and the triangle inequality, we can compute the maximum error as $(x[s], y[s])$ approximates (x,y):

$$\begin{aligned}
|(x[s], y[s]) - (x,y)| &\leq |(x[s], y[s]) - (r_s\cos\theta_s, r_s\sin\theta_s)| + \\
&\qquad |(r_s\cos\theta_s, r_s\sin\theta_s) - (x,y)| \\
&\leq 2^{-s} + |(r_s\cos\theta_s, r_s\sin\theta_s) - (r\cos\theta, r\sin\theta)| \\
&\leq 2^{-s} + M|(r_s,\theta_s) - (r,\theta)| \\
&\leq 2^{-s} + M\sqrt{(r-r_s)^2 + (\theta-\theta_s)^2} \\
&< 2^{-s} + M\sqrt{(2)2^{-2s}} \\
&= 2^{-s}(1 + M\sqrt{2})
\end{aligned}$$

⊣

Since $2^{-t} = 2^{-s}(1+M\sqrt{2}) \iff t = s - \log\left(1 + M\sqrt{2}\right)$, we know $(x[s], y[s])$ computes (x,y) at precision $t = s - \log(1+M\sqrt{2})$. Letting $\Delta t = \log(1+M\sqrt{2})$, we hence have

$$K_{s-\Delta t}(x,y) \leq K(x[s], y[s]) \leq K(\overline{x}[s]\overline{y}[s]) + c'$$

where c' is the machine constant turning $\overline{x}[s]$ and $\overline{y}[s]$ into the real approximations $x[s]$ and $y[s]$. Since $t + \Delta t = s$, the right-hand side of Lemma 3.2 implies

$$\begin{aligned}
K_s(x,y) &\leq K_{s-\Delta t}(x,y) + K(t) + c_2(\Delta t) + c \\
&\leq K(\overline{x}[s]\overline{y}[s]) + c' + K(t) + c_2(\Delta t) + c
\end{aligned}$$

where $c_2(\Delta t)$ is as in Lemma 3.2 and hence does not depend on t or s.

We require a final claim:

Claim 2. $K(\overline{x}[s]\overline{y}[s]) \leq K(\overline{r}[s]\overline{\theta}[s]) + c''$ *for some constant* c''.

Proof of Claim 2. By definition, $r_s = r[s]$ and $\theta_s = \theta[s]$. Then note that $x[s] = (r_s\cos\theta_s)[s]$, which is computable via approximations and Taylor's Theorem, with some machine constant c''. The same holds for $y[s]$. ⊣

Combining our previous bound with the claim, we obtain

$$\begin{aligned} K_s(x,y) &\le K(\overline{x}[s]\overline{y}[s]) + c' + K(t) + c_2(\Delta t) + c \\ &\le K(\overline{r}[s]\overline{\theta}[s]) + c' + c'' + K(t) + c_2(\Delta t) + c. \end{aligned}$$

Recall that $\Delta t = \log(1+M\sqrt{2})$ is constant. Further, $t = s - \Delta t$, and thus there exists a constant c''' (which only depends on Δt and M, and not on s) for which $K(t) = K(s-\Delta t) \le K(s) + c'''$. Hence, combining constants in d,

$$K_s(x,y) \le K(\overline{r}[s]\overline{\theta}[s]) + K(s) + d. \qquad \square$$

For the second half, we make the following brief observation. The argument below will focus on points in $\mathbb{D}$ that do not lie on the first axis; this is necessary for a bounding argument involving Lipschitz conditions: for each point (x, y) not on the first axis, we can find a nice neighbourhood on which the coordinate transformation map from Euclidean to polar coordinates is nicely behaved. What about the points *on* the first axis? There is nothing to do, for if $x \ge 0$ then the polar coordinates and Euclidean coordinates of the point $(x, 0)$ coincide. Hence Proposition 3.1 holds on the first axis trivially.

Lemma 3.5 (Second half of Proposition 3.1)**.** *There exists c such that if $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) then there exist $N_{(x,y)}, \Delta$ such that if $s > N_{(x,y)}$ then*

$$K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]) \le K_s(x,y) + K(s) + c.$$

Proof. First, we make an approximating observation.

Claim 1. *If $a \in \mathbb{Q}^2 \cap B_{2^{-r}}(x,y)$ then*

$$(x[r], y[r]) \in B_{2^{-r}(1+\sqrt{2})}(a).$$

Proof of Claim 1. By assumption, $|(x, y) - a| < 2^{-r}$, so by the triangle inequality we have

$$\begin{aligned} |(x[r], y[r]) - a| &\le |(x[r], y[r]) - (x,y)| + |(x,y) - a| \\ &= \sqrt{(x[r]-x)^2 + (y[r]-y)^2} + |(x,y)-a| \\ &\le \sqrt{2\left(2^{-2r}\right)} + 2^{-r} \\ &\le 2^{-r}\sqrt{2} + 2^{-r} \\ &= 2^{-r}(1+\sqrt{2}) \end{aligned} \qquad \dashv$$

In the notation of [36], let $\mathcal{Q}^2_r = \{2^{-r}z \mid z \in \mathbb{Z}^2\}$ denote the set of r-dyadics. Note that r-dyadics have at most r-many non-zero post-binary-point bits. We bound the number of r-dyadics in an open ball:

Claim 2. *For any $a \in \mathbb{Q}^2$ and $r < \omega$, we have*

$$|\mathcal{Q}^2_r \cap B_{2^{-r}(1+\sqrt{2})}(a)| \le \left(4\left(1+\sqrt{2}\right)\right)^2.$$

Proof of Claim 2. Let C_2 be the square centred at a with side length given by $2(1+\sqrt{2})2^{-r}$. Since $B_{2^{-r}(1+\sqrt{2})} \subset C_2$, we have

$$|\mathcal{Q}^2_r \cap B_{2^{-r}(1+\sqrt{2})}| \le |\mathcal{Q}^2_r \cap C_2|.$$

Observe that C_2 has area $(2(1+\sqrt{2}))^2 2^{-2r}$. Now, if $x, y \in \mathcal{Q}_r^2$ and $x \neq y$ then we have $|x-y| \geq 2^{-r}$ (since the elements in $\mathcal{Q}_r^2$ have at most r-many non-zero post-binary-point bits). Hence consider a **small square**: a square of side length 2^{-r}. Such a small square has area 2^{-2r} and cannot contain more than 4 r-dyadics: one on each of its vertices. Hence, dividing the area of C_2 by the area of a small square and multiplying by 4 for each vertex gives an upper bound for the number of r-dyadics:

$$\begin{aligned}|\mathcal{Q}_r^2 \cap B_{2^{-r}(1+\sqrt{2})}| &\leq \frac{\left(2\left(1+\sqrt{2}\right)\right)^2 2^{-2r}}{2^{-2r}} 2^2 \\ &= \left(2\left(1+\sqrt{2}\right)\right)^2 2^2 \\ &= \left(4\left(1+\sqrt{2}\right)\right)^2\end{aligned}$$

$\dashv$

Let M with program P be a machine that: on input $\pi = \pi_1\pi_2\pi_3$ if $h(\pi_1) = \overline{k}$ with $k < \omega$, and $h(\pi_2) = \overline{s}$ with $s < \omega$, and $h(\pi_3) = \overline{a}$ with $a \in \mathbb{Q}^2$, then M outputs the k-th dyadic rational in $B_{2^{-s}(1+\sqrt{2})}(a)$. Now, if $a \in \mathbb{Q}^2$ is such that $K(a) = K_s(x, y)$, then the claims together imply that there exists some $k < (4(1+\sqrt{2}))^2$ for which $(x[s], y[s])$ is the k-th element in the intersection $\mathcal{Q}_s^2 \cap B_{2^{-s}(1+\sqrt{2})}(a)$. Let π_1, π_2, π_3 be witnesses for $K(k)$, $K(s)$ and $K(a)$, respectively. Then

$$h\left(0^{\ell(P)}1P\pi_1\pi_2\pi_3\right) = \overline{x}[s]\overline{y}[s]$$

and thus

$$\begin{aligned}K(\overline{x}[s]\overline{y}[s]) &\leq \ell(\pi_1) + \ell(\pi_2) + \ell(\pi_2) + c \\ &= K(k) + K(s) + K_s(x, y) + c' \\ &\leq K_s(x, y) + K(s) + c\end{aligned}$$

where $K(k)$ can be bounded above by a constant.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the computable function mapping a point in Euclidean coordinates to its polar coordinates. On $\mathbb{D}$ (excluding the first axis), this map is given by

$$(x, y) \mapsto \left(\sqrt{x^2+y^2}, \tan^{-1}(y/x)\right)$$

and is continuously differentiable. Let $\epsilon > 0$ such that the closed ball B of radius ϵ centred at (x, y) does not intersect the first axis. By Lemma 3.3, f is Lipschitz on B. Now, suppose $s < \omega$ is such that $2^{-s} < \epsilon$ (thus $B_{2^{-s}}(x, y) \subset B$), and let $(p, q) \in \mathbb{Q}^2 \cap B_{2^{-s}}(x, y)$. Since $f(x, y) = (r, \theta)$ we have

$$\begin{aligned}|(r, \theta) - f(p, q)| &= |f(x, y) - f(p, q)| \\ &\leq M|(x, y) - (p, q)| \\ &\leq M2^{-s} \\ &= 2^{-(s-\log M)}.\end{aligned}$$

Thus, the machine that computes f (with machine constant c'', by claim 2 of the proof of Lemma 3.4) yields $(x[s], y[s])$, which yields the polar coordinates (r, θ) up to precision $s - \log M$:

$$\begin{aligned}K(\overline{r}[s-\log M]\overline{\theta}[s-\log M]) &\leq K(\overline{x}[s]\overline{y}[s]) + c'' \\ &\leq K_s(x, y) + K(s) + c'' + c\end{aligned}$$

$\square$

Proof of Proposition 3.1. The proof is now a consequence of the previous two lemmas and the following claim:

Claim 1. *If* $\Delta < \omega$ *then* $|K(\overline{r}[s]\overline{\theta}[s]) - K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta])| \leq c$ *for some constant* c.

Proof of Claim 1. Computing $\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]$ from $\overline{r}[s]\overline{\theta}[s]$ is straightforward. For the other direction, let $\overline{r}(\Delta)$ be such that $\overline{r}[s] = \overline{r}[s-\Delta]\overline{r}(\Delta)$, and equally for $\overline{\theta}$. Suppose that

$$\begin{aligned} h(\pi_1) &= \overline{r}[s-\Delta]\overline{\theta}[s-\Delta] \\ h(\pi_2) &= \overline{r}(\Delta) \\ h(\pi_3) &= \overline{\theta}(\Delta) \end{aligned}$$

are all optimal. Let p be a program that on input $\pi = \pi_1\pi_2\pi_3$, merges the two strings obtained by π_2 and π_3 with the string from π_1 in the obvious way (cf. Section 2.5). Then

$$\overline{r}[s]\overline{\theta}[s] = h\left(0^{\ell(p)}1p\pi_1\pi_2\pi_3\right)$$

and thus optimality of programs implies

$$\begin{aligned} K(\overline{r}[s]\overline{\theta}[s]) &\leq \ell(\pi_1) + \ell(\pi_2) + \ell(\pi_3) + c \\ &= K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]) + K(\overline{r}(\Delta)) + K(\overline{\theta}(\Delta)) + c. \end{aligned}$$

Note that $\ell(\overline{r}(\Delta)) = \Delta$, and recall $K(\sigma) \leq \ell(\sigma) + 2\log(\ell(\sigma)) + c'$ for all $\sigma \in 2^{<\omega}$. Since Δ is fixed,

$$\begin{aligned} K(\overline{r}[s]\overline{\theta}[s]) \leq K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]) &+ \\ 2\ell(\overline{\Delta}) + 4\log(\ell(\overline{\Delta})) &+ c. \end{aligned} \quad \dashv$$

The claim now yields the result from the previous two lemmas: let $(x, y) \in \mathbb{D}$ with polar coordinates (r, θ). Suppose Δ is as in Lemma 3.5. Then

$$\begin{aligned} \dim(r,\theta) &= \liminf_{s\to\infty} \frac{K_s(r,\theta)}{s} \\ &= \liminf_{s\to\infty} \frac{K(\overline{r}[s]\overline{\theta}[s])}{s} \\ &= \liminf_{s\to\infty} \frac{K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta])}{s} \\ &= \liminf_{s\to\infty} \frac{K_s(x,y)}{s} \\ &= \dim(x,y) \end{aligned}$$

since $K(s) \leq \log(s) + 2\log(\log(s) + 1) + c$ for a constant c. □

Below, we pass to polar coordinates as required. In particular, the points of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets we build in Theorems 4.1 and B will be determined by their radii, which we construct explicitly.

From now on, if we write (r, θ) below, we usually mean the point with Euclidean coordinates $(r\cos\theta, r\sin\theta)$; in such cases, $(r, \theta) \in \mathbb{D}$. We occasionally return to Euclidean coordinates, and we explicitly mention when we do so.

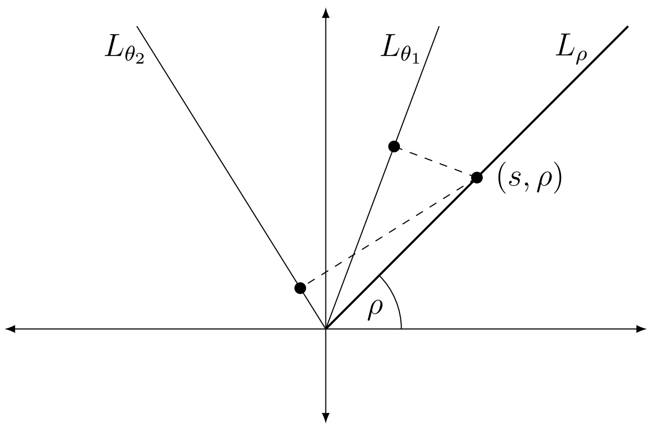

FIGURE 2. The projections of (s,ρ) onto L_{θ_1} and L_{θ_2} are computed directly.

3.2. **Projections in Polar Coordinates.** We now focus on the behaviour of projections of sets onto straight lines. Consider $\theta \in [0,\pi]$, and let L_θ be the straight line that passes through the origin at angle θ with the first coordinate axis. It is clear that $[0,\pi]$ exhausts all straight lines through the origin. Let $(s,\rho) \in \mathbb{D}$ and denote by $\operatorname{proj}_\theta(s,\rho)$ the **projection of** (s,ρ) **onto** L_θ: the unique point of intersection of L_θ with the unique perpendicular-to-L_θ line containing (s,ρ). Recall that if $(s,\rho) \in \mathbb{D}$ then $0 \le s \le 1$ (cf. figs. 2 and 3).

There are two cases: $|\theta-\rho| \le \pi/2$ or $|\theta-\rho| > \pi/2$. If we have $|\theta-\rho| \le \pi/2$ then $|\operatorname{proj}_\theta(s,\rho)| = s\cos(\theta-\rho)$; otherwise, $|\operatorname{proj}_\theta(s,\rho)| = s\cos((\theta+\pi)-\rho)$. Since $0 \le s \le 1$ and $\cos(x+\pi) = -\cos(x)$, we conclude:

Lemma 3.6. *For every $(s,\rho) \in \mathbb{D}$ and every $\theta \in [0,\pi]$ we have*

$$|\operatorname{proj}_\theta(s,\rho)| = s|\cos(\theta-\rho)|.$$

In particular, the polar coordinates of the projection of (s,ρ) onto L_θ are

$$\operatorname{proj}_\theta(s,\rho) = \begin{cases} (s|\cos(\theta-\rho)|,\theta) & \text{if } |\theta-\rho| \le \pi/2 \\ (s|\cos(\theta-\rho)|,\theta+\pi) & \text{otherwise.} \end{cases}$$

Now suppose $E \subseteq \mathbb{D}$ and fix some $\theta \in [0,\pi]$. Define

$$E(\theta) = \{s|\cos(\theta-\rho)| \mid (s,\rho) \in E\} \subset \mathbb{R}. \tag{3.2}$$

We show below that, in fact, $\dim_H(E(\theta)) = \dim_H(\operatorname{proj}_\theta(E))$.

We need the following notions: a real number $x \in \mathbb{R}$ is **computable** if there exists a machine that uniformly on input $k < \omega$ outputs a rational $q \in \mathbb{Q}$ such that $q \in B_{2^{-k}}(x)$; this naturally extends to $\mathbb{R}^m$ for $m \ge 1$.

Lemma 3.7. *If $x \in \mathbb{R}^m$ is computable then* $\dim(x) = 0$.

Proof. Suppose M with program p is a machine that on input $\overline{s}$ for $s < \omega$ computes $\overline{q_s}$ for some $q_s \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$. Then $h\left(0^{\ell(p)}1p\overline{s}\right) = \overline{q_s} \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$ and so $K_s(x) \le \ell(\overline{s}) + c$. Recall that $\ell(\overline{s}) \le \log(s)+1$ and thus

$$\dim(x) \le \liminf_{s\to\infty} \frac{\log(s)+1+c}{s} = 0. \qquad \square$$

Lemma 3.8. *If $E \subset \mathbb{R}^2$ is countable, then* $\dim_H(E) = 0$.

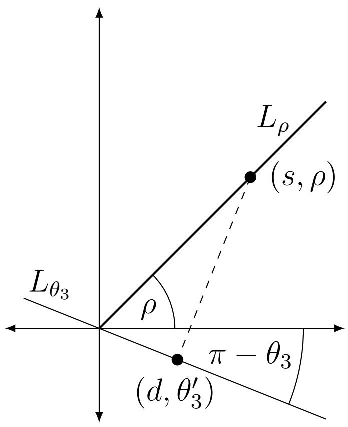

FIGURE 3. For L_{θ_3}, the projection lies in the fourth quadrant. There, $\theta_3' = \pi - \theta_3$, and so $d = s\cos(\rho + \pi - \theta_3) = s|\cos(\theta_3 - \rho)|$.

Proof. Suppose $E = \{x_i \mid i < \omega\}$, and let $X = \bigoplus \overline{x_i}$, the infinite join. Let M with program p be a machine with oracle to access to X that on input $(\bar{i}, \bar{s})$ computes $\overline{x_i}[s]$. Then M computes all x_i, and hence by Lemma 3.7 and the Point-to-Set Principle 2.9 we have $\dim_H(E) \leq \sup_{x \in E} \dim^X(x) = 0$. □

Lemma 3.9. *Let $r \in \mathbb{R}$. Then for every oracle $A \in 2^\omega$ the following hold.*

(1) $\dim^A(r) = \dim^A(r, 0)$
(2) $\dim^A(r) = \dim^A(-r)$

Proof. Modulo constants, we have

$$K(\bar{r}[s]) \leq K\left(\bar{r}[s]\bar{0}[s]\right) \leq K(\bar{r}[s]) + K\left(\bar{0}[s]\right).$$

Since 0 is computable, Lemma 3.7 implies

$$\lim_{s \to \infty} \frac{K\left(\bar{0}[s]\right)}{s} = 0.$$

Applying lim inf yields item 1. For item 2, easily computes $\overline{-r}[s]$ from $\bar{r}[s]$. Both arguments relativise. □

Lemma 3.10. *Let $\theta \in [0, \pi)$. If $E \subseteq \mathbb{D}$ then*

$$\dim_H(\text{proj}_\theta(E)) = \dim_H(E(\theta)).$$

Proof. Fix $\theta \in [0, \pi]$ and suppose $(s, \rho) \in \mathbb{D}$. For brevity, define

$$p(s, \rho) = |\text{proj}_\theta(s, \rho)| = s|\cos(\theta - \rho)|$$

by Lemma 3.6. Now, item 1 of Lemma 3.9 implies

$$\dim^A(p(s, \rho)) = \dim^A(p(s, \rho), 0)$$

for every oracle $A \in 2^\omega$. Hence, let

$$P_\theta(E) = \{(p(s, \rho), 0) \mid (s, \rho) \in E\} \subset \mathbb{R}^2.$$

Note that $\dim_H(E(\theta)) = \dim_H(P_\theta(E))$ by the Point-to-Set Principle 2.9. We now aim to appeal to Corollary 2.2: $\dim_H$ is invariant under rotations. However, rotating $P_\theta(E)$ by θ anti-clockwise does not necessarily yield $\text{proj}_\theta(E)$: if there exists $(s, \rho) \in E$ with $|\theta - \rho| > \pi/2$ then $\text{proj}_\theta(s, \rho) = (p(s, \rho), \theta + \pi)$, not $(p(s, \rho), \theta)$.

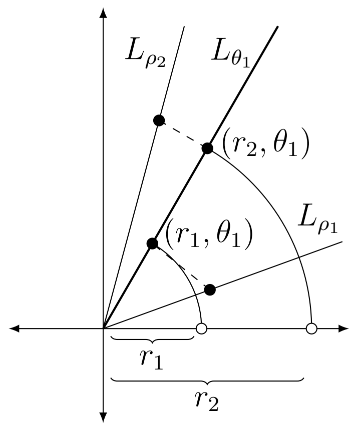

FIGURE 4. If $|\rho - \theta| \leq \pi/2$, consider the length of the projections on the first axis, and rotate.

Hence: whenever $(s, \rho) \in E$ and $|\theta - \rho| > \pi/2$, passing to Euclidean coordinates, consider $(-p(s, \rho), 0)$ instead. Let

$$p^*(s, \rho) = \begin{cases} p(s, \rho) & \text{if } |\theta - \rho| \leq \pi/2 \\ -p(s, \rho) & \text{otherwise.} \end{cases}$$

and define $P^*_\theta(E)$ (cf. figs. 4 and 5) in Euclidean coordinates:

$$P^*_\theta(E) = \{(p^*(s, \rho), 0) \,|\, (s, \rho) \in E\}$$

By items 1 and 2 of Lemma 3.9, it follows that

$$\dim^A(p(s, \rho), 0) = \dim^A(p^*(s, \rho), 0)$$

for all oracles $A \in 2^\omega$. Hence, the Point-to-Set Principle 2.9 implies

$$\dim_H(P_\theta(E)) = \dim_H(P^*_\theta(E)).$$

Moreover, rotating $P^*_\theta(E)$ by θ yields $\text{proj}_\theta(E)$. Hence, Corollary 2.2 implies

$$\begin{aligned} \dim_H(E(\theta)) &= \dim_H(P_\theta(E)) \\ &= \dim_H(P^*_\theta(E)) \\ &= \dim_H(\text{proj}_\theta(E)). \end{aligned}$$ □

We complete this section with an effective proof to a useful result in geometric measure theory. To our knowledge, this result has not yet appeared in print.

Lemma 3.11. *If $E \subseteq \mathbb{R}^2 \setminus \{0\}$ intersects every line through the origin in $\mathbb{D}$, then*

$$\dim_H(E) \geq 1.$$

Proof. Suppose $A \in 2^\omega$, and let $B \in 2^\omega$ be A-random. Thus, $\overline{\theta} = 0001B \in 2^\omega$ codes a real number $\theta \in (0, 1)$. Since B is A-random, by definition of relativised dimension, $K^A(B \upharpoonright s) \geq s - c$ for some constant c. As $B \upharpoonright s$ is easily computed from $\overline{\theta}[s]$, we have

$$s - c \leq K^A(B \upharpoonright s) \leq K^A(\overline{\theta}[s]) + c'$$

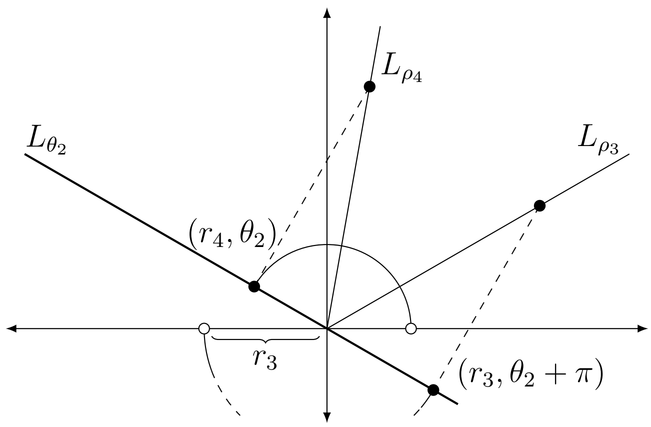

FIGURE 5. If $|\rho - \theta| > \pi/2$, mirror along the second axis and then rotate by θ.

for some machine constant c'. Thus

$$\begin{aligned}\dim^A(\theta) &= \liminf_{s\to\infty} \frac{K^A(\overline{\theta}[s])}{s}\\ &\geq \liminf_{s\to\infty} \frac{K^A(B \upharpoonright s)}{s}\\ &\geq \liminf_{s\to\infty} \frac{s-c}{s} = 1.\end{aligned}$$

As E intersects L_θ, there exists $r > 0$ such that $(r, \theta) \in E$. Note that we can easily compute $\overline{\theta}[s]$ from $\overline{r}[s]\overline{\theta}[s]$, so

$$K^A(\overline{\theta}[s]) \leq K^A(\overline{r}[s]\overline{\theta}[s]) + c''$$

for some machine constant c''. Hence,

$$\begin{aligned}\dim^A(r,\theta) &= \liminf_{s\to\infty} \frac{K^A(\overline{r}[s]\overline{\theta}[s])}{s}\\ &\geq \liminf_{s\to\infty} \frac{K^A(\overline{\theta}[s])}{s}\\ &= \dim^A(\theta) = 1.\end{aligned}$$

Since A was arbitrary, the proof is complete. □

3.3. **Constructing $\mathbf{\Pi}^1_1$ Sets by Recursion.** To build $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, we use a method originally due to Erdős, Kunen, and Mauldin [13] and A. Miller [43], and recently generalised by Z. Vidnyánszky [58]. Assuming $V{=}L$, this **$\underset{\sim}{\mathbf{\Pi}}^1_1$-Recursion Theorem** allows us to construct sets of reals recursively in the style of Ciesilski [8], Davies [10], and others, with the added benefit that the constructed set is $\underset{\sim}{\mathbf{\Pi}}^1_1$. While the $\underset{\sim}{\mathbf{\Pi}}^1_1$-Recursion Theorem is very general (it applies to all computably presentable Polish spaces), we focus on a special case.

For notational simplicity, if $X = \{x_\alpha \mid \alpha < \omega_1\}$ we define the **truncation of X to α** by $X \upharpoonright \alpha = \{x_\beta \mid \beta < \alpha\}$.

Definition 3.12 ($V{=}L$)**.** Let $F \subset \mathbb{D}^{\leq\omega} \times [0, \pi/2] \times \mathbb{D}$. Then $X = \{x_\alpha \,|\, \alpha < \omega_1\}$ is **compatible with F** if $X \subset \mathbb{D}$ and there exist:

- an enumeration $\{p_\alpha \,|\, \alpha < \omega_1\}$ of $[0, \pi/2]$; and
- a set $\{A_\alpha \,|\, \alpha < \omega_1\} \subset \mathbb{D}^{\leq\omega}$ for which $A_\alpha = X \upharpoonright \alpha$ for every $\alpha < \omega_1$

such that $(A_\alpha, p_\alpha, x_\alpha) \in F$ for every $\alpha < \omega_1$.

Observe that since $A_\alpha \in \mathbb{D}^{\leq\omega}$, each A_α has order type $\leq \omega$.

Definition 3.13. A set $X \subset 2^\omega$ is **cofinal in the Turing degrees** if it is cofinal in the partial ordering of Turing degrees. If $m \geq 1$ and $X \subset \mathbb{R}^m$ then X is cofinal in the Turing degrees if the set $\{\overline{x} \mid x \in X\}$ is.

We now state the **$\undertilde{\Pi}^1_1$-Recursion Theorem**, which in full generality is due to Z. Vidnyánszky [58, Thm. 1.3].

Theorem 3.14 ((V=L), The $\undertilde{\Pi}^1_1$-Recursion Theorem)**.**
Suppose $F \subset \mathbb{D}^{\leq\omega} \times [0, \pi/2] \times \mathbb{D}$. If F is $\undertilde{\Pi}^1_1$ and if for all $(A, p) \in \mathbb{D}^{\leq\omega} \times [0, \pi/2]$ the section $F(A, p) = \{x \in \mathbb{D} \mid F(A, p, x)\}$ is cofinal in the Turing degrees, then there exists a $\undertilde{\Pi}^1_1$ set compatible with F.

Theorem 3.14 proves a recursion principle: at stage α we have access to A_α (the set of elements we have already enumerated into X) and to the current **condition** to be satisfied, p_α. The section $F(A_\alpha, p_\alpha)$ now gives the set of **candidates** which both satisfy condition p_α and respect A_α. Since $A_\alpha = X \upharpoonright \alpha$, we know that X satisfies all conditions.

In our proofs of Theorem 4.1 and of Theorem B, the set of candidates is $\mathbb{D}$. Conditions are straight lines through the origin, indexed by $[0, \pi/2]$.

4. The Proof of Theorem A

In this section, we prove our first main result.

Theorem A. ZFC *does not prove* MP *for all $\undertilde{\Pi}^1_1$ subsets of $\mathbb{R}^2$.*

To prove Theorem A, we assume V=L and construct a $\undertilde{\Pi}^1_1$ set which fails MP. This is the content of Theorem 4.1, which we prove in this section.

Theorem 4.1 (V=L)**.** *There exists a $\undertilde{\Pi}^1_1$ set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while $\dim_H(\mathrm{proj}_\theta(E)) = 0$ for all $\theta \in [0, 2\pi)$.*

4.1. Roadmap Towards a Proof.

Assume V=L. Let $B = \{\theta_\alpha \mid \alpha < \omega_1\}$ be an enumeration of $[0, \pi/2]$. We argue by induction on ω_1 and hence build $E \subset \mathbb{D}$ satisfying Theorem 4.1 in stages; we think of the angles in B as the **conditions** (or **requirements**) which need to be satisfied. During our construction, when considering condition φ, we also handle $\varphi + \pi/2$ simultaneously. By Theorem 3.14, at stage α we have access to all points (r_i, θ_i) already enumerated into E. We satisfy condition θ_α. Denote $\theta = \theta_\alpha$, and argue thus:

(1) Let $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\}$, the set of points already enumerated into E. For each $i < \omega$ the angular coordinate θ_i tells us which conditions are already satisfied.
(2) Build $r \in (0, 1)$ such that for all $i < \omega$:
  - $\dim(r|\cos(\theta - \theta_i)|) = 0$
  - $\dim(r|\cos(\theta + \pi/2 - \theta_i)|) = 0$

  This suffices by Lemma 3.10.
(3) Enumerate the pair (r, θ) into E.

Note that the set of reals in item (2) must be cofinal in the Turing degrees for the $\undertilde{\Pi}^1_1$-Recursion Theorem 3.14 to apply. The following result is essential.

Proposition 4.2. *Suppose $a_i \in (0,1)$ for all $i < \omega$. There exists $r \in (0,1)$ such that $\dim(a_i r) = 0$ for all $i < \omega$. The set of such r is cofinal in the Turing degrees.*

We prove Proposition 4.2 in Section 4.2. Having it in hand, we may already prove Theorem 4.1. We need one more lemma.

Lemma 4.3. *If $A \in 2^\omega$ and $a \in \mathbb{R}$, then $\{x \in \mathbb{R} \mid \dim^A(x) = a\}$ is Borel.*

Proof. The subsequent arguments relativise. By definition,

$$\dim(x) = \liminf_{n\to\infty} \frac{K_n(x)}{n} = \liminf_{n\to\infty} \frac{\min\{K(q) \mid q \in \mathbb{Q} \cap B_{2^{-n}}(x)\}}{n}.$$

Since the lim inf of a sequence of Borel functions is Borel, it suffices to show that

$$K_n(x) = \min\{K(q) \mid q \in \mathbb{Q} \cap B_{2^{-n}}(x)\}$$

is Borel. To see this, note that

$$\begin{aligned} K_n(x) < c &\iff \exists q \in \mathbb{Q} \cap B_{2^{-n}}(x)(K(q) < c) \\ &\iff x \in \bigcup_{q \in \mathcal{K}(c)} B_{2^{-n}}(q) \end{aligned}$$

where $\mathcal{K}(c) = \{p \in \mathbb{Q} \mid K(p) < c\}$. Hence K_n is Borel, and so is dim. □

Proof of Theorem 4.1. To use the $\underset{\sim}{\mathbf{\Pi}}^1_1$-Recursion Theorem 3.14, we define the set $F \subset \mathbb{D}^{\leq\omega} \times [0, \pi/2] \times \mathbb{D}$ by

$$\begin{gathered} (A, \varphi, (r, \theta)) \in F \text{ if and only if} \\ \varphi = \theta \text{ and for all } (r', \theta') \in \operatorname{ran}(A) \text{ we have} \\ \dim(r|\cos(\varphi - \theta')|) = \dim(r|\cos(\varphi + \pi/2 - \theta')|) = 0. \end{gathered}$$

By Lemma 4.3, F is $\underset{\sim}{\mathbf{\Pi}}^1_1$.

Note φ is satisfied by a point on L_φ. Let $\varphi \in [0, \pi/2]$, and focus on the sections of F: for $\alpha < \omega_1$, by definition,

$$F(A, \varphi) = \{(r, \theta) \mid (A, \varphi, (r, \theta)) \in F\}.$$

Suppose $A = \{(r_i, \theta_i) \mid i < \omega\} \in \mathbb{D}^{\leq\omega}$. Let

$$a_i = |\cos(\varphi - \theta_i)| \quad \text{and} \quad b_i = |\cos(\varphi + \pi/2 - \theta_i)|.$$

By construction,

$$(r, \theta) \in F(A, \varphi) \iff \theta = \varphi \text{ and } \dim(ra_i) = \dim(rb_i) = 0 \text{ for all } i < \omega.$$

Proposition 4.2 implies that this section is cofinal in the Turing degrees. Therefore, using Lemma 4.3, we may apply the $\underset{\sim}{\mathbf{\Pi}}^1_1$-Recursion Theorem 3.14: there exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set

$$E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\} \subset \mathbb{R}^2$$

which is compatible with F. We also have enumerations $\{\varphi_\alpha \mid \alpha < \omega_1\} = [0, \pi/2]$ and $\{A_\alpha \mid \alpha < \omega_1\}$ such that $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\} = E \upharpoonright \alpha$ and for each $\alpha < \omega_1$,

$$(r_\alpha, \theta_\alpha) \in F(A_\alpha, \varphi_\alpha).$$

Note that we have $\theta_\alpha = \varphi_\alpha$.

We verify that E is as required. Let $\varphi \in [0, \pi]$. We show $\dim_H(\mathrm{proj}_\varphi(E)) = 0$. By Lemma 3.10, it suffices to show that $\dim_H(E(\varphi)) = 0$ (where we define $E(\varphi) = \{r|\cos(\varphi - \theta)| \mid (r, \theta) \in E\}$ as in eq. (3.2)). We show this below.

Note that there exist $\delta < \omega_1$ and $\varphi_\delta \in [0, \pi/2]$ such that:

- either $\varphi = \varphi_\delta$,
- or $\varphi = \varphi_\delta + \pi/2$.

Let δ be such, and recall $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$. We consider the points that were enumerated *before* condition φ_δ and those enumerated *after* φ_δ separately.

$\leq \delta$: At condition φ_δ, define (analogous to Lemma 3.8) the oracle

$$X = \bigoplus \left\{ \overline{r_\beta |\cos(\varphi_\delta - \theta_\beta)|}, \overline{r_\beta |\cos(\varphi_\delta + \pi/2 - \theta_\beta)|} \;\middle|\; \beta \leq \delta \right\}.$$

Then X computes both $r_\beta|\cos(\varphi_\delta - \theta_\beta)|$ and $r_\beta|\cos(\varphi_\delta + \pi/2 - \theta_\beta)|$ for all $\beta \leq \delta$. Since

$$\varphi \in \{\varphi_\delta, \varphi_\delta + \pi/2\}$$

Lemma 3.7 implies that for all $\beta \leq \delta$:

$$\dim^X(r_\beta|\cos(\varphi - \theta_\beta)|) = 0.$$

$> \delta$: We show that $\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0$ for every $\beta > \delta$. Let $\delta < \beta < \omega_1$. Then we have

$$(r_\beta, \theta_\beta) \in F(A_\beta, \varphi_\beta) = F(E \upharpoonright \beta, \varphi_\beta).$$

But the conditions we have already attended to at stage β are exactly the angular coordinates of the points in $E \upharpoonright \beta$; in particular,

$$E \upharpoonright \beta = \{(r_\alpha, \varphi_\alpha) \mid \alpha < \beta\}.$$

So, for all $\gamma < \beta$, again by definition of F, we have

$$\dim(r_\beta|\cos(\varphi_\gamma - \theta_\beta)|) = \dim(r_\beta|\cos(\varphi_\gamma + \pi/2 - \theta_\beta)|) = 0.$$

Since $\delta < \beta$ and either $\varphi = \varphi_\delta$ or $\varphi = \varphi_\delta + \pi/2$, we have in particular

$$\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0.$$

We picked $\delta < \beta < \omega_1$ arbitrarily, hence this holds for all such β.

Thus, by the Point-to-Set Principle 2.9 and Lemma 3.10,

$$\begin{aligned}
\dim_H(\mathrm{proj}_\varphi(E)) &= \dim(E(\varphi)) \\
&= \min_{A \in 2^\omega} \sup_{\alpha < \omega_1} \dim^A(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\
&\leq \sup_{\alpha < \omega_1} \dim^X(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\
&= 0.
\end{aligned}$$

Now $\dim_H(E) \geq 1$ by Lemma 3.11. □

Finally, the fact that $\dim_H(E) = 1$ is a consequence of the following lemma.

Lemma 4.4. *Suppose $E \subset \mathbb{D}$. Then $\dim_H(\mathrm{proj}_\theta(E)) \geq \dim_H(E) - 1$.*

Proof. Suppose $(r, \theta) \in \mathrm{proj}_\theta(E)$. By Lemma 3.6, we know that $r = s|\cos(\theta - \rho)|$ for some $(s, \rho) \in E$. But now, note that, given (r, θ), we can compute s from ρ, and vice versa. Hence suppose $\dim(r, \theta) = \epsilon$. Since $\dim(s), \dim(\rho) \leq 1$ we see that $\dim(s, \rho) \leq \dim(r, \theta) + 1$, which is as desired. □

Since the set E constructed in Theorem 4.1 is $\mathbf{\Pi}^1_1$ and fails property MP, we have hence shown our first main theorem:

Theorem A. ZFC *does not prove* MP *for all $\mathbf{\Pi}^1_1$ subsets of $\mathbb{R}^2$.*

As a result, we obtain sharpness:

Corollary 4.5. *Assuming* ZFC*, the regularity property* MP *is sharp for* $\underset{\sim}{\mathbf{\Sigma}}^1_1$ *subsets of* $\mathbb{R}^2$*:* ZFC *proves* MP *for all* $\underset{\sim}{\mathbf{\Sigma}}^1_1$ *sets, but not for all* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *subsets.*

4.2. **Proving Proposition 4.2.** To prove this technical proposition, we introduce additional notation. An interval is called **(open) dyadic** if it is of the form $(j/2^k,(j+1)/2^k)$. Intervals of the form $[j/2^k,(j+1)/2^k]$ are **closed dyadic**. Observe that

$$x \in (j/2^k,(j+1)/2^k) \implies |x - j/2^k| \leq 2^{-k}.$$

Hence, x and $j/2^k$ agree on the first k bits in their binary expansion: both start with the binary expansion of j.

Below, we work with open intervals in $(0,1)$. All reals are expressed in binary. Instead of manipulating intervals directly, we argue in terms of dyadic reals. Let $\sigma \in 2^{<\omega}$ and $\tau \in 2^{\leq\omega}$.

- Let $\tilde{\tau} = 0.\tau \in \mathbb{R}$.
- Let $\tilde{\sigma}_+ = 0.\sigma 1^\infty \in \mathbb{R}$.
- Let $[\tilde{\sigma}]$ denote the open interval $(\tilde{\sigma},\tilde{\sigma}_+)$.
- If $a \in \mathbb{R}$ then let $a[\tilde{\sigma}] = (a\tilde{\sigma}, a\tilde{\sigma}_+)$.

Basic facts that follow by definition are summarised below.

Lemma 4.6. *Let* $\sigma,\rho \in 2^{<\omega}$*. Suppose* I *is dyadic.*

(1) The interval $[\tilde{\sigma}]$ *is dyadic. If* $x \in [\tilde{\sigma}]$ *then* x *and* $\tilde{\sigma}$ *agree on the initial segment of length* $\ell(\sigma)$*. (We say* x **extends** σ*.)*
(2) If $\tilde{\sigma}$ *is the left-end point of* I*, then* $I = [\tilde{\sigma}]$*.*
(3) If $\tilde{\sigma} \in I$*, then* $[\tilde{\sigma}] \subset I$*.*
(4) If $\rho \in 2^{<\omega}$ *then* $\sigma \prec \rho$ *if and only if* $\tilde{\rho} \in [\tilde{\sigma}]$*.*

Lemma 4.7. *Suppose* $\sigma \in 2^{<\omega}$ *and* $a,\epsilon \in (0,1)$*. There exist* $\rho,\tau \in 2^{<\omega}$ *such that* $\sigma \prec \rho$*,* $a[\tilde{\rho}] \subset [\tilde{\tau}]$*, and* $K(\tau)/\ell(\tau) < \epsilon$*.*

Proof. Let σ, a and ϵ be given. Consider $a[\tilde{\sigma}]$. Since $[\tilde{\sigma}]$ is open, so is $a[\tilde{\sigma}]$, and thus it contains a closed dyadic interval. Take the largest (in diameter) such interval I, and pick $\tau' \in 2^{<\omega}$ such that $\tilde{\tau}'$ is the left end-point of I. By closedness, $\tilde{\tau}' \in a[\tilde{\sigma}]$. By standard results on Kolmogorov complexity, there exists a least $s < \omega$ such that $\tau = \tau' 0^s$ satisfies

$$\frac{K(\tau)}{\ell(\tau)} < \epsilon.$$

In particular, $\tilde{\tau}' = \tilde{\tau} \in I$. Now, since $[\tilde{\tau}]$ is open so is $a^{-1}[\tilde{\tau}]$. Let J denote the largest closed dyadic interval contained in $a^{-1}[\tilde{\tau}]$, and call its left end-point d. Again by closedness, $d \in a^{-1}[\tilde{\tau}]$. Let $\rho \in 2^{<\omega}$ be such that $\tilde{\rho} = d$.

Now $\sigma \prec \rho$: by construction, $\tilde{\rho} = d \in J \subset a^{-1}[\tilde{\tau}]$. The string τ properly extends τ', thus $[\tilde{\tau}] \subset [\tilde{\tau}']$. Since $\tilde{\tau}'$ is the left end-point of I, the interior of I equals $[\tilde{\tau}']$. Hence, we have $[\tilde{\tau}] \subset [\tilde{\tau}'] \subset I \subset a[\tilde{\sigma}]$, and so $\tilde{\rho} \in a^{-1}[\tilde{\tau}] \subset a^{-1}(a[\tilde{\sigma}]) = [\tilde{\sigma}]$, as needed. We also have $a[\tilde{\rho}] \subset [\tilde{\tau}]$, since $[\tilde{\rho}] \subset J \subset a^{-1}[\tilde{\tau}]$. ☐

To achieve cofinality in the Turing degrees when constructing $r \in (0,1)$, we code a given oracle $A \in 2^\omega$ into r while satisfying each condition (as per item (2) in Section 4.1). Let

$$\nu(k) = 2^{2^k}$$

determine where to code A in r. We use the intervals in $\mathrm{ran}(\nu)$ to satisfy our conditions. We call ν the **folding map**.

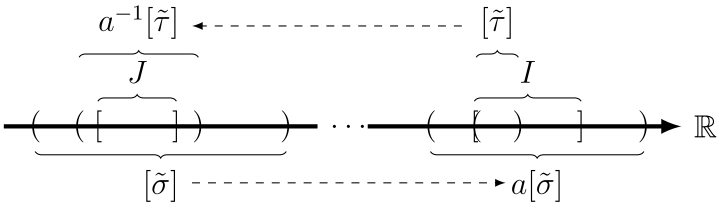

FIGURE 6. We start on the left and argue anti-clockwise: $a[\tilde{\sigma}]$ is an open interval; the largest closed dyadic interval inside is I. Picking a suitable $\tilde{\tau} \in I$ yields $a^{-1}[\tilde{\tau}]$. The largest dyadic interval contained in it is J with left end-point $d = \tilde{\rho}$. Hence, $[\tilde{\rho}] \subset J$, where in fact the interior of J equals $[\tilde{\rho}]$.

4.2.1. *The Construction of* r. Suppose (a_i) is the set of conditions, where for every $i < \omega$ we have $a_i \in (0,1)$. We construct $r \in (0,1)$ in stages, by determining its binary expansion, which is given by successive extensions $x_0 \prec x_1 \prec x_2 \prec \dots$ with $x_i \in 2^{<\omega}$. We argue by induction on ω.

(1) Let $A \in 2^\omega$ be given.
(2) Let $x_0 = \emptyset$, the empty string.
(3) Let x_k be given. At stage $k+1$, decode $k+1 = \langle i, n\rangle$ via Cantor's pairing function, for instance, and attend to requirement i. Hence we attend to each requirement infinitely often.
(4) Apply Lemma 4.7 with $a = a_i$ and $\epsilon = \frac{1}{k}$ to obtain a suitable extension $\rho_k \succ x_k$.
(5) Let $t = \nu(k+1) - \ell(\rho_k) - 1$ and $d = A(k)$ and define

$$x_{k+1} = \begin{cases} \rho_k 0^t d & \text{if } \ell(\rho_k) < \nu(k+1) \\ (\rho_k \restriction (\nu(k+1)-1))d & \text{otherwise.} \end{cases} \tag{4.1}$$

Therefore, if $k > 0$ then $\ell(x_k) = \nu(k)$ by induction.
(6) Define $x = \bigcup_{k<\omega} x_k$, and let $r = \tilde{x}$.

Note that A is computably **coded in** x: $x(\nu(k+1)-1) = A(k)$ for all $k < \omega$. To complete the proof of Proposition 4.2, we must ensure that the second case in eq. (4.1) only occurs finitely often for each requirement a_i. This is the content of the next lemma.

Lemma 4.8. *For each $a_i \in (0,1)$ there exists $M_i < \omega$ such that if $k+1 > M_i$ and $k+1 = \langle i, n\rangle$ attends to requirement a_i, then $\ell(\rho_k) < \nu(k+1)$.*

Let $(x,y) \subseteq (0,1)$. We will use the following facts.

(i) Denote the diameter of (x,y) by $\mathrm{diam}((x,y)) = y - x$. If $\sigma \in 2^{<\omega}$ then $\mathrm{diam}([\tilde{\sigma}]) = 2^{-\ell(\sigma)}$. In particular, we have $-\log(\mathrm{diam}([\tilde{\sigma}])) = \ell(\sigma)$.
(ii) If $k < \omega$ is such that $k \geq -\log(\mathrm{diam}((x,y))) + 2$ then there exists $j < \omega$ such that $[j/2^k, (j+1)/2^k] \subset (x,y)$.

Proof of Lemma 4.8. Fix $a_i = a$ and suppose we are at stage $k+1 = \langle i, n\rangle$. Let $\rho = \rho_k$. Recall that $\tilde{\rho} \in J \subset a^{-1}[\tilde{\tau}]$. Note $\mathrm{diam}(a^{-1}[\tilde{\tau}]) = a^{-1}2^{-\ell(\tau)}$. Since J is defined to be the maximal closed dyadic interval in $a^{-1}[\tilde{\tau}]$, and since $\tilde{\rho}$ is the left end-point of J, items (i) and (ii) and the fact that $\tau = \tau' 0^s$ imply

$$\begin{aligned}\ell(\rho) &\leq -\log(\operatorname{diam}(a^{-1}[\tilde{\tau}])) + 2 \\ &= \log(a) - \log\left(2^{-\ell(\tau)}\right) + 2 \\ &= \log(a) + \ell(\tau) + 2 \\ &= \log(a) + \ell(\tau') + s + 2.\end{aligned}$$

Recall that $\rho \succ x_k$ (so $x_k = \sigma$ in Lemma 4.7). By construction, $\tilde{\tau}' \in I \subset a[\tilde{x}_k]$, where I is dyadic maximal in $a[\tilde{x}_k]$. So,

$$\ell(\tau') \leq -\log(\operatorname{diam}(a[\tilde{x}_k])) + 2 = -\log(a) + \ell(x_k) + 2$$

from which we obtain via item (5) that

$$\ell(\rho) \leq \ell(x_k) + s + 4 = \nu(k) + s + 4.$$

At stage $k{+}1$, we build $x_{k+1} \succ x_k$ where $\ell(x_{k+1}) = \nu(k{+}1)$. Note that our construction is successful if we do not truncate ρ (as in the second case in eq. (4.1)). Then, we have

$$\ell(\rho) < \ell(x_{k+1}) = \nu(k+1).$$

Hence, it suffices to show that we have $s < \nu(k+1) - \nu(k) - 4$ for large enough k.

Recall that s satisfies $\frac{K(\tau)}{\ell(\tau)} = \frac{K(\tau' 0^s)}{\ell(\tau')+s} < \frac{1}{k}$, and simplify:

$$\begin{aligned}\frac{K(\tau' 0^s)}{\ell(\tau') + s} &\leq \frac{K(\tau') + K(0^s) + c'}{s} \\ &\leq \frac{K(\tau')}{s} + \frac{K(s)}{s} + \frac{c''}{s} \\ &\leq \frac{\ell(\tau') + 2\log(\ell(\tau'))}{s} + \frac{\log(s) + 2\log(\log(s)+1)}{s} + \frac{c}{s}\end{aligned}$$

for a sum of machine constants c. These terms are easily bounded: $\frac{c}{s} < \frac{1}{3k}$ if $s > 3kc$. For the middle term, $\log(s) + 2\log(\log(s)+1) < 3\log(s)$ once $s \geq 2$. Hence,

$$\frac{\log(s) + 2\log(\log(s)+1)}{s} < \frac{3\log(s)}{s}.$$

Since $\log(s)/s$ is monotonically decreasing, if $s > 2^k$ then

$$\frac{3\log(s)}{s} < \frac{3\log\left(2^k\right)}{2^k} = \frac{3k}{2^k}.$$

Then $\frac{3k}{2^k} < \frac{1}{3k}$ if $9k^2 < 2^k$ which holds for $k \geq 10$. Hence, for large enough k, the bound $s > 2^k$ suffices. For the first term, recall that $\ell(\tau') \leq -\log(a) + \nu(k) + 2$. Since $a \in (0,1)$ we know $-\log(a) > 0$. So, for large enough k, it follows that

$$\begin{aligned}\frac{\ell(\tau') + 2\log(\ell(\tau'))}{s} &\leq \frac{-\log(a) + \nu(k) + 2}{s} + \frac{2\log(-\log(a) + \nu(k) + 2)}{s} \\ &\leq \frac{-\log(a) + 3\nu(k)}{s}\end{aligned}$$

Since a is fixed we have, for large enough k, that

$$\frac{\ell(\tau') + 2\log(\ell(\tau'))}{s} \leq \frac{-\log(a) + 3\nu(k)}{s} \leq \frac{4\nu(k)}{s}.$$

Observe that $\frac{4\nu(k)}{s} \leq \frac{1}{3k}$ if $s > 12k\nu(k)$. Choosing k large enough, we see that $s > \max\{3kc, 2^k, 12k\nu(k)\}$ suffices, which also reduces to $s > 12k\nu(k)$ once k is large enough.

Finally, $12k\nu(k) + 1 < \nu(k+1) - \nu(k) - 4$ for $k \geq 3$. Thus, if k satisfies the conditions above, then $s = 12k\nu(k) + 1$ satisfies $\frac{K(\tau' 0^s)}{s} < \frac{1}{k}$ while we also have $s < \nu(k+1) - \nu(k) - 4$. So, eventually, $\ell(\rho)$ is sufficiently small. □

Proof of Proposition 4.2. Fix $A \in 2^\omega$, and suppose (a_i) is the countable sequence of requirements. Construct $x = \bigcup_{k<\omega} x_k$ as in Section 4.2.1. Let $r = \tilde{x}$. From Section 2.5 and Definition 3.13, A can be computed from the binary expansion of r. Hence, we only show that $\dim(a_i r) = 0$. Fix $i < \omega$. By Lemma 4.8, there exists M such that if $k = \langle i, n\rangle > M$ then $\rho_k \prec x$. For each k, let τ_k be as in Lemma 4.7 alongside ρ_k. By construction, $a_i[\tilde{\rho}_k] \subset [\tilde{\tau}_k]$ and $a_i\tilde{\rho}_k \in [\tilde{\tau}_k]$. Thus, we have $a_i r = a_i \tilde{x} \in [\tilde{\tau}_k]$. Further, $K(\tau_k)/\ell(\tau_k) < 1/k$. Letting

$$D = \{k > M \mid k = \langle i, n\rangle \text{ for some } n\}.$$

we apply Corollary 2.11 to obtain

$$\begin{aligned}\dim(a_i r) &\leq \liminf_{k\to\infty,\, k\in D} \frac{K(\overline{a_i r}[\ell(\tau_k)])}{\ell(\tau_k)}\\ &\leq \liminf_{k\to\infty,\, k\in D} \frac{K(\tau_k) + c}{\ell(\tau_k)}\\ &\leq \liminf_{k\to\infty,\, k\in D} \frac{1}{k}\\ &= 0\end{aligned}$$

where c is the machine constant obtaining τ_k from $\overline{a_i r}[\ell(\tau_k)]$. □

5. The Proof of Theorem B

In this section, we prove a significant extension of Theorem 4.1 which settles the provability of MP in terms of the geometric measure theoretic complexity of counterexamples.

Theorem B. *For every* $\epsilon \in [0,1]$ *there exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E_\epsilon \subset \mathbb{R}^2$ *such that we have* $\dim_H(E_\epsilon) = 1 + \epsilon$ *while* $\dim_H(\mathrm{proj}_\theta(E_\epsilon)) = \epsilon$ *for every* $\theta \in [0, 2\pi)$.

Observe that choosing $\epsilon = 0$ in Theorem B yields Theorem 4.1. The case $\epsilon = 1$ is trivial: if $E \subseteq \mathbb{R}^2$ satisfies $\dim_H(E) = 2$ then Lemma 4.4 implies

$$1 \leq \dim_H(\mathrm{proj}_\theta(E)) \leq 1.$$

for each θ. Hence, Theorem B is exhaustive and optimal.

5.1. Roadmap Towards a Proof.

Let $0 < \epsilon < 1$. Assuming $V{=}L$, we argue as follows.

1. Fix an enumeration $\{\varphi_\alpha \mid \alpha < \omega_1\}$ of $[0, \pi/2]$.
2. At stage α, let $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\}$, the set of all points already enumerated into our set.
3. Let $X \in 2^\omega$ be the sequence whose bits are made up of the binary expansion of φ_α. In particular, X is $\overline{\varphi}_\alpha$ with its first four bits removed (cf. Section 2.5).

(4) We will *not* satisfy condition φ_α by enumerating a point on L_{φ_α} into our set. Instead, we recover the already satisfied conditions by first coding them into r using a suitable folding map: if (r_i, θ_i) was enumerated into our set at stage β, then φ_β is folded into r_i, and can hence be recovered computably. Let $\{\varphi_i \mid i < \omega\}$ be the set of the conditions already satisfied.
(5) Pick $\theta \in [0, \pi/2]$ such that $\overline{\theta}$ is random relative to X.
(6) Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$. Let Y be the join of X and all $\overline{a}_i$. We also assume that Y computes ϵ.
(7) Construct $r \in (0, 1)$ such that:
   (a) the binary expansion of φ_α is folded computably into the binary expansion of r;
   (b) $\dim(ra_i) = \epsilon$ for all $i < \omega$;
   (c) $\dim^{Y,\overline{\theta}}(r) = \epsilon$.

We will give some insight into the verification below. Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$.

In our construction of a suitable r, we adapt the methods used in the proof of Theorem 4.1. However, instead of inserting long strings of zeroes into the binary expansions of ra_i, we pick a good oracle $T \in 2^\omega$ and fold it into ra_i.

An oracle T is **suitable** if it is random relative to $Y \oplus \overline{\theta}$ (and hence all $\overline{a}_i$). Now, assume r is constructed. Then Y (which computes all $\overline{a}_i$) computes initial segments of T from initial segments of r: by computing an initial segment of ra_i for the correct i. Since T is random relative to $Y \oplus \overline{\theta}$, we force $\dim^{Y,\overline{\theta}}(r)$ to not drop too low by coding T not too sparsely.

The details can be found in Section 5.5, and the theorem then follows by Lemma 4.4. Further, we use the following simple result, which follows from symmetry of information.

Lemma 5.1. *Let $A \in 2^\omega$ be an oracle. For any $x, y \in \mathbb{R}$ we have*

$$\dim^A(x, y) \geq \dim^A(x) + \dim^{A,\overline{x}}(y).$$

For an in-depth account of the interplay between relativised dimension and **conditional dimension** of elements of $\mathbb{R}^n$ see Lutz and Lutz [33, 4.3 and 4.4], who introduced the latter notion ibidem. The previous lemma is also a consequence of their arguments, and a proof can be found there, too; it follows from the fact that $K(x \mid y) \geq K^{\overline{y}}(x)$.

Recall that Y computes X, and hence $\dim^X(r) \geq \dim^Y(r)$ for all r. Thus,

$$\dim^X(r, \theta) \geq \dim^X(\theta) + \dim^{X,\overline{\theta}}(r) \geq \dim^X(\theta) + \dim^{Y,\overline{\theta}}(r) \geq 1 + \epsilon$$

since $\overline{\theta}$ is random relative to X, and by our construction of r. The final steps of this high-level verification are then as follows: suppose we construct E broadly as in the proof of Theorem 4.1. By the same argument as in said proof, The Point-to-Set Principle 2.9 implies

$$\dim_H(\text{proj}_\theta(E)) = \dim_H(E(\theta)) \leq \epsilon$$

since allowing oracles can only decrease the dimension of points. conversely, every oracle X appears as some $\varphi_\alpha \in [0, \pi/2]$. Hence, there exists $(r_\alpha, \theta_\alpha)$ for which $\overline{\theta}$ is random relative to X. Since such a point exists for *every* oracle, the Point-to-Set Principle 2.9 implies

$$\dim_H(E) \geq \dim^X(r_\alpha, \theta_\alpha) \geq \dim^X(\theta) + \dim^{X,\overline{\theta}}(r) \geq 1 + \epsilon$$

by Lemma 5.1. Then the conclusion follows from Lemma 4.4.

5.2. **Folding Oracles Into r.** Fix $\epsilon \in (0,1)$, and let

$$Z = Y \oplus \overline{\theta}$$

recalling that Y already computes all $\overline{a}_i$. Instead of constructing $T \in 2^\omega$ random relative to Z and then coding it sparsely to obtain dimension ϵ, we use a result due to Athreya, Hitchcock, Lutz, and Mayordomo [1, Thm. 6.5]:

Lemma 5.2. *Let $0 \leq \alpha \leq 1$. There exists $x \in \mathbb{R}$ such that*

$$\dim(x) = \operatorname{Dim}(x) = \alpha.$$

The authors obtain this real precisely by sparsely coding a random sequence, interleaved with strings of zeroes. Their result relativises, and their construction shows that one may assume that

$$\dim^Z(\overline{x}) = \dim(\overline{x}) = \epsilon.$$

For our purposes, letting $T = \overline{x}$ for a suitable x given by Lemma 5.2 suffices.

In the construction of Theorem 4.1 we focused on satisfying requirements: we demanded a particular number of consecutive zeroes to appear in the image in order to push the complexity down sufficiently far; and in our verification, we showed that, eventually, the gap between conditions will be sufficiently large so that enough zeroes can be accommodated. Here, we must take more care; we must always be able to give a good bound on the number of computable bits of T. Hence, we fix the number of bits to be appended so that there is no “overspill”. The following lemma yields such a bound.

Lemma 5.3. *In the argument of Lemma 4.7:*

$$s = \nu(k+1) - \nu(k) - 5 \implies \ell(\rho_k) < \nu(k+1).$$

Proof. This follows from the proof of Lemma 4.8: with a, ρ, τ' as there,

$$\ell(\rho_k) \leq \log(a) + \ell(\tau') + s + 2 \leq \nu(k) + s + 4.$$

Equating this to $\nu(k+1)$ and demanding strict inequalities yields the proof. $\square$

In particular, if we have space for s bits, we can code $s-5$ bits into the image. This leads to the following corollary:

Corollary 5.4. *In the proof of Lemma 4.7, with $a \in (0,1)$: if $\ell(\rho) = m$ and $n > m$ then if $s = n - m - 5$ we have $\ell(\rho') < n$, where ρ' is the extension of ρ that codes s bits into $r\tilde{\rho}'$.*

We now choose our folding map ν to be

$$\nu(k) = 2^{2^k} + k.$$

We introduce the shift summand k so as to make sure that the gaps between $\nu(k)$ and $\nu(k+1)$ have length

$$2^{2^{k+1}} + k + 1 - 2^{2^k} - k = 2^{2^{k+1}} - 2^{2^k} + 1$$

where the last bit is reserved to code a bit of φ_α into ra_i (as per item 7a). The gap we have to extend is exactly of length

$$\nu(k+1) - \nu(k) - 1 = 2^{2^{k+1}} - 2^{2^k} = 2^{2^k}\left(2^{2^k} - 1\right) \tag{5.1}$$

which is divisible by $2^{(2^k - k)}$. This fact will be useful below.

5.3. **Coding and Saving Blocks.** Naïvely, our arguments ought to work as follows: at each stage, we construct a radius r that, together with a suitable angle θ, satisfies the active requirement. To preserve the high dimension of r (relative to Z) and of the points ra_i, we code segments of T into each ra_i. In particular: if a_j is attended to right after a_i, and the last bit of T coded into ra_i is $T(k)$ for some $k < \omega$, then the first bit of T coded into ra_j at that stage is $T(k+1)$. Hence, with a long enough initial segment of r, the oracle Z can compute a long initial segment of T by picking the correct a_i (which Z computes), computing ra_i, and picking out the coded bits of T.

For ease of notation, suppose $T \in 2^\omega$ and define

$$T^{(m,n)} = \langle T(m), T(m+1), \dots, T(n-1) \rangle.$$

In particular, observe that $\ell(T^{(m,n)}) = n-m$, and that $T(n)$ does not appear in $T^{(m,n)}$. The dimension of ra_i is bounded above by the dimension of T: taking a sufficiently long initial segment $\overline{ra_i}[t]$ of ra_i, we easily find a long string of the form $T^{(m,n)}$ coded into it. If $\ell(T^{(m,n)}) = n - m$ is large enough compared to t, then dimension will decrease—this requirement is ensured by choosing a sparse enough folding map.

However, it is now difficult to show that the dimension of ra_i does not drop properly below the dimension of T. The problem is that it is in general hard to tell how many bits in the multiplication of reals are determined by a single bit: e.g. if $a = 1/\pi$ and $\tilde{\sigma} = 0.\sigma$ for some $\sigma \in 2^{<\omega}$, and $\tau \succ \sigma$, there is no bound on how many bits of the product $a\tilde{\tau}$ are correct in the sense that every extension yields the same initial segment.

We circumvent this issue as follows: as we extend r, we **save blocks** of bits that are coded into ra_i throughout the stage. We do this by pulling back the interval, as seen in Lemma 4.7. Hence we define the **block map** $\mu\colon \omega \to \omega$ by

$$\mu(k) = 2^{(2^k - k)}.$$

Recall that our folding map is given by $\nu(k) = 2^{2^k} + k$. Hence, at stage k with r_k at hand, we have $\nu(k+1) - \nu(k)$ many bits to extend r_k. In particular, the *number of blocks fitting into the gap of stage* $k+1$ is given by

$$\xi(k) = \frac{\nu(k+1) - \nu(k) - 1}{\mu(k)} = \frac{2^{2^k}\left(2^{2^k} - 1\right)}{2^{(2^k - k)}} = 2^k\left(2^{2^k} - 1\right). \tag{5.2}$$

Note that $\xi(k)\mu(k) = \nu(k+1) - \nu(k) - 1$.

A few lemmas are needed.

Firstly, we need to have a good bound on how many bits we can code into ra_i at each stage k, and in each block. And secondly, it is not clear that saving blocks does not cost too many bits. The first is not an issue due to Corollary 5.4. We resolve the second later in the **Cost Lemma** 5.5, after introducing the construction in detail.

As we code T in blocks, it is prudent to describe a suitable partitioning of T beforehand. By recursion, reconstruct T into segments T_k^j, where k denotes the last completed stage (so if we see T_k^j then we are in stage $k+1$), and j the active block. At stage $k+1$: (1) we code $\xi(k) = 2^k\left(2^{2^k} - 1\right)$-many blocks, which follows from eq. (5.2); and (2) each block of T coded into the image has length $\mu(k) - 5$, as we lose 5 bits each time as per Corollary 5.4. Hence,

$$T = \bigcup_{3 \le k < \omega} \left(\bigcup_{1 \le j \le \xi(k)} T_k^j \right)$$

where the union operator denotes concatenation. Hence

$$T = T_3^1 \cup T_3^2 \cup \ldots \cup T_3^{2040} \cup T_3^1 \cup \ldots T_k^{\xi(k)} \cup T_{k+1}^1 \ldots$$

as $\xi(3) = 2040$. As mentioned, we lose 5 bits each time we code a T-block, thus

$$\ell(T_k^j) = \mu(k) - 5 = 2^{(2^k - k)} - 5.$$

An easy calculation shows that 3 is least to allow coding of bits, which is why the outer union starts at $k = 3$. In particular, the first stage at which bits are coded is stage $k + 1 = 4$, with $\ell(T_3^j) = 27$ and $\xi(3) = 2040$.

5.4. **The Construction.** Recall the folding map $\nu(k) = 2^{2^k} + k$ and the block map $\mu(k) = 2^{(2^k - k)}$. The radius r is now constructed as follows: suppose φ_α is the next condition.

(1) Let $A \in 2^\omega$.
(2) Let $x_0 = \emptyset$, the empty string.
(3) Let x_k be given. At stage $k + 1$, decode $k + 1 = \langle i, n \rangle$; we now attend to requirement i.
(4) We iterate over all $\xi(k)$-many blocks.
Let $0 \le j < \xi(k) = 2^k(2^{2^k} - 1)$.
  (a) Let $x_k^0 = x_k$.
  (b) At block $j+1$, x_k^j is given. We apply Lemma 4.7 and code T_k^{j+1} into $a_i \tilde{x}_k^j$. Let ρ_k^{j+1} be the resulting extension. By filling up with s-many zeroes (via Lemma 5.3), we find $x_k^{j+1} = \rho_k^{j+1} 0^s$ of length

$$\ell(x_k^j) + \mu(k) = \ell(x_k) + 2^{(2^k - k)}(j + 1).$$

(5) After the last block, we have one bit left to code A or $\overline{\varphi}_\alpha$ (this follows from eq. (5.1)). By construction, we have $\ell\left(x_k^{\xi(k)}\right) = \nu(k+1) - 1$; hence define

$$x_{k+1} = x_k^{\xi(k)} d$$

where

$$d = \begin{cases} A(k/2) & \text{if } k \text{ is even} \\ \overline{\varphi}_\alpha((k-1)/2) & \text{if } k \text{ is odd.} \end{cases}$$

(6) Define $x = \bigcup_{k<\omega} x_k$, and let $r = \tilde{x}$.

Observe that $\ell(x_{k+1}) = \nu(k+1)$, and that we code the active line into r. Further, we code A as in Theorem 4.1 to apply the $\underset{\sim}{\mathbf{\Pi}}_1^1$-Recursion Theorem 3.14. This completes the construction.

5.5. **The Verification.** Below, we prove that $\dim(ra_i) \le \epsilon$ (Section 5.5.1); and that $\dim^Z(r) \ge \epsilon$, where $Z = Y \oplus \overline{\theta}$ (Section 5.5.2). Then Theorem B follows from Lemma 4.4.

5.5.1. *The Dimension of ra_i.* Both verification arguments are "bit counting" arguments: we exhibit a piece of a complicated string coded inside ra_i, and show that it is long enough in a precise sense: its length dwarves the length of all non-coded bits. Let $a = a_i$ and consider $\overline{ar}[m]$ for some m such that

$$\overline{ar}[m] = \sigma \cup \left(\bigcup_{1 \leq j \leq \xi(k)} \sigma_j T_k^j \right)$$

for some k; hence stage $k+1$ has just been completed. (Considering the strings at the end of stages is prudent as we easily have access to a long consecutive segment of T, albeit interrupted.) We also know by Lemma 4.8 that

$$\ell(\sigma) \leq -\log(a) + \ell(x_k) + 2 = -\log(a) + \nu(k) + 2.$$

Further, the **cost** of saving a block is given by a bound on the length of each σ_j.

Lemma 5.5 (The Cost Lemma)**.** *Let $a \in (0,1)$ and $r_m \in 2^{<\omega}$. As in Lemma 4.7, find $\tilde{\tau}_m$ and I_m dyadic such that $[\tilde{\tau}_m] \subset I_m \subset a[\tilde{r}_m]$; let τ'_m be the left end-point of I_m. Further, let $J \subset a^{-1}[\tilde{\tau}_m]$ be dyadic, where $\tilde{\rho}_k$ is the left-endpoint of J. Let $r_{m+1} = \rho_m 0^t$ so that $\ell(r_{m+1}) = \ell(r_m) + \mu(k)$ where k denotes the current stage. Suppose τ'_{m+1} is the left end-point of $I_{m+1} \subset a[\tilde{r}_{m+1}]$.*
Then $|\ell(\tau'_{m+1}) - \ell(\tau_m)| \leq 7$.

Before we proceed with the proof, a few comments are in order. Firstly, consulting fig. 6 alongside the statement and proof of the above lemma is useful, as it serves as its motivation. Conceptually, one thinks of the hypotheses as the intermediate step between moving from one block to the next within a given stage in our construction: r_m is the available string in block m inside some stage, and $\rho \succ \sigma$ is its computed extension. Importantly, $a[\tilde{r}_{m+1}]$ contains τ_m as a substring. We ask: after saving τ_m in $a[\tilde{r}_{m+1}]$, how many bits are lost before we begin coding the next block? In particular, if we construct a real r by such approximations r_m and we have established that

$$ar \succ \tau_m \lambda \tau_{m+1}$$

for some $\lambda \in 2^{<\omega}$ by successive block saving, then how long can λ be at most?

Proof. By assumption we have $[\tilde{\tau}_m] \subset I_m \subset [\tilde{r}_m]$, and so

$$\operatorname{diam}([\tilde{\tau}_m]) \leq \operatorname{diam}(I_m) \leq \operatorname{diam}([\tilde{r}_m]).$$

Applying $-\log$ and by item (i) we have

$$-\log(\operatorname{diam}(I_m)) \in [-\log(a) + \ell(r_k), \ell(\tau_k)].$$

Since $\tilde{\tau}'_k$ is the left end-point of I_k we also have

$$\ell(\tau'_m) \in [-\log(a) + \ell(r_m), \ell(\tau_m)].$$

For a better bound, use item (ii) from page 22 to obtain

$$\ell(\tau'_m) \leq -\log(\operatorname{diam}(a[\tilde{r}_m])) + 2 = -\log(a) + \ell(r_m) + 2$$

and hence

$$\ell(\tau'_m) \in [-\log(a) + \ell(r_m), -\log(a) + \ell(r_m) + 2].$$

At stage $k+1$, Corollary 5.4 implies that each block has $\mu(k)-5$ bits coded into its image. Hence $\ell(\tau_m) = \ell(\tau'_m) + (\mu(k)-5)$. Therefore, observing by construction that $\ell(\tau'_{m+1}) \geq \ell(\tau_m)$,

$$\begin{aligned}\ell(\tau'_{m+1}) - \ell(\tau_m) &= \ell(\tau'_{m+1}) - \ell(\tau'_m) - (\mu(k)-5) \\ &\leq -\log(a) + \ell(r_{m+1}) + 2 + \log(a) - \ell(r_m) - (\mu(k)-5) \\ &= (\ell(r_{m+1}) - \ell(r_m)) - (\mu(k)-5) + 2 \\ &= \mu(k) - (\mu(k)-5) + 2 \\ &= 7\end{aligned}$$

where we use that the block size is $\mu(k)$, and hence $\ell(r_{m+1}) - \ell(r_m) = \mu(k)$. □

So, $\ell(\sigma_j) \leq 7$. For simplicity, we let

$$T_k = T_k^1 \cup \ldots \cup T_k^{\xi(k)};$$

hence $\ell(T_k) = \xi(k)(\mu(k)-5)$. The next lemma provides the final technical detail in this half of our verification. For simplicity of notation, let

$$S_k = \bigcup_{1\leq j\leq \xi(k)} \sigma_j T_k^j.$$

Lemma 5.6. *For $k < \omega$ and $\sigma, (\sigma_j)$ as above, we have*

$$|K(T_k) - K(\sigma S_k)| \leq O(2^{2^k}).$$

Proof. This is another "bit counting" argument: the number of bits by which T_k and σS_k differ is given by $\ell(\sigma) + \sum_j \ell(\sigma_j)$. If we also know where the σ_j's are located, then we can construct each string from the other. Thus,

$$|K(T_k) - K(\sigma S_k)| \leq K(\sigma) + \sum_{1\leq j\leq \xi(k)} K(\sigma_j, m_j)$$

omitting constants, where m_j denotes the index at which σ_j begins inside S_k. We know $\ell(\sigma_j) \leq 7$ and

$$\ell(\sigma) \leq -\log(a) + \ell(x_k) + 2 = -\log(a) + 2^{2^k} + k + 2.$$

Further,

$$\begin{aligned}\ell(\sigma S_k) &= \ell(\sigma) + \sum_{1\leq j\leq \xi(k)} \ell(\sigma_j) + \ell(T_k) \\ &\leq -\log(a) + \ell(x_k) + 2 + 7\xi(k) + \xi(k)(\mu(k)-5) \\ &= -\log(a) + \ell(x_k) + 2 + \xi(k)(\mu(k)+2)\end{aligned}$$

since each of the $\xi(k)$-many blocks codes $\mu(k)-5$-many bits. Observe that

$$\xi(k)\mu(k) = 2^{2^k}(2^{2^k}-1)$$

and hence is of order $2^{2^{k+1}}$. As $m_j \leq \ell(S_k)$ we see that m_j is thus at most of order $2^{2^{k+1}}$. But now $K(m_j)$ is at most of order 2^{k+1}. It is now readily seen that $\sum_j K(\sigma_j, m_j)$ is of order at most $\xi(k)2^{k+1}$, which is $O(2^{2^k})$. □

Using Lemma 5.6, we now see

$$K(\overline{ar}[m]) = K(\sigma S_k) = K(T_k) + O(2^{2^k}).$$

Further, observe that $\ell(T_k)$ is of order $2^{2^{k+1}}$, since $\ell(T_k) = \xi(k)(\mu(k) - 5)$. As before, $\lim_{k\to\infty} \frac{2^{2^k}}{2^{2^{k+1}}} = 0$, and so we may ignore terms of order at most 2^{2^k}. So we simplify: let $\mathcal{D}$ be the set of $m < \omega$ at which requirement $a = a_i$ has just been attended (in other words, $\overline{ar}[m] = \sigma S_k$ for some k). Then, by definition of T,

$$\dim(ar) \leq \liminf_{m\in\mathcal{D}} \frac{K(\overline{ar}[m])}{m} \leq \liminf_{m\in\mathcal{D}} \frac{K(T_k)}{m} = \epsilon$$

5.5.2. *The Z-Dimension of r.* Recall that $Z = Y \oplus \overline{\theta}$ and that Y computes all a_i. As we must show $\dim^Z(r) \geq \epsilon$, it does not suffice to exhibit a set of favourable elements, such as our set $\mathcal{D}$ above. Instead, we show how to decode enough elements of T from *any* initial segment of r. Suppose we are at stage $k+2$ and n blocks have already been satisfied. Write

$$\overline{r}[m] = \sigma_1 \cdots \sigma_{k+1} b_1 \cdots b_n \tau$$

where

- σ_i denotes the initial segment of r that satisfied stage i;
- b_j denotes the substring of r that satisfied block j of stage $k+2$; and
- τ is the initial segment satisfying block $n+1$.

At stage $k+1$, the substring T_k has been coded into ar. Hence, using the oracle Z which computes all a_i, we can recover T_k from $\overline{ar}$. Recall that

$$\ell(T_k) = \xi(k)(\mu(k) - 5).$$

Observe that since $\lim_{k\to\infty} \frac{2^{2^k}}{2^{2^{k+1}}} = \lim_{k\to\infty} \frac{1}{2^k} = 0$, the length of T_k already dwarves the lengths of $T_1 + \ldots + T_{k-1}$; hence, it suffices to compute the blocks saved at stage $k+1$.

The worst case to consider above is the case $n = 0$: then, the initial segment σ_{k+1} needs to carry enough information to survive against τ, where we have $\ell(\tau) \leq \mu(k+1) - 1$. This is not an issue, since $\ell(T_k) = \xi(k)(\mu(k) - 5)$ and

$$\lim_{k\to\infty} \frac{\mu(k+1) - 1}{\xi(k)(\mu(k) - 5)} = 0.$$

Hence, the information provided in T_k dwarves the unfinished block τ. It now suffices to show that T_k and the completely coded substrings $T^1_{k+1}, \ldots, T^n_{k+1}$ can be recovered from $\overline{ar}[m]$. This follows from the argument of Lemma 5.6:

- Take a machine that trims $\overline{r}$ to length $\nu(k+1) - 1$, and denote the resultant string by ρ (this is where stage $k+1$ has just been completed).
- Compute the projection factor $a_i = a$ for stage $k+1$ using Z (and from the Cantor pairing function).
- Compute the largest dyadic interval in $a[\tilde{\rho}]$, and let d denote its left end-point. Now, we have $\overline{d} = \sigma S_k \sigma'$ where $\ell(\sigma') \leq 7$, by the Cost Lemma 5.5.

By Lemma 5.6, the complexity of isolating T_k from $\sigma S_k \sigma'$ is not significant, as required. An identical argument recovers the n blocks. It also follows from Lemma 5.6 that

$$K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1}) \leq K^Z(\overline{r}[m]) + O(2^{2^k}) + O(n2^{k+2})$$

where $n < \xi(k+1)$. Thus, in particular

$$\frac{K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})}{m} \leq \frac{K^Z(\overline{r}[m])}{m} + \frac{O(2^{2^k}) + O(n2^{k+2})}{m} \tag{5.3}$$

where $m = \nu(k+1) + n\mu(k+1) + \ell(\tau)$ and $n < \xi(k+1)$. Next, we verify that the length of T computed on the left-hand side of eq. (5.3) is sufficiently long: note that

$$|m - \ell(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})| = \ell(\tau) + \nu(k) + 5\xi(k) + 5n + 1.$$

Now, $m = \nu(k+1) + n\mu(k+1) + \ell(\tau)$ and $n < \xi(k+1)$ imply

$$\frac{\ell(\tau) + \nu(k) + 5\xi(k) + 5n}{m} \leq \frac{\ell(\tau) + \nu(k) + 5(\xi(k) + \xi(k+1))}{\ell(\tau) + \nu(k+1) + \xi(k+1)\mu(k+1)}.$$

Applying limits as k goes to infinity shows that the term vanishes. Applying lim inf to both sides of eq. (5.3) yields

$$\liminf_{k\to\infty} \frac{K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})}{m} = \epsilon.$$

Finally, since m is of order $\nu(k+1) + n\mu(k+1)$, i.e. of order at least $2^{2^{k+1}}$, the right-hand side of eq. (5.3) simplifies to its first term. Hence

$$\epsilon = \liminf_{k\to\infty} \frac{K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})}{m} \leq \frac{K^Z(\overline{r}[m]}{m} = \dim^Z(r).$$

Theorem B now follows from the same arguments as in Theorem 4.1, and the overview given at the start of this section.

6. Open Questions

We list open question for future investigation.

By a theorem of P. Mattila [40], Marstrand's Projection Theorem can be extended to higher dimensions in Euclidean space. Whether the optimal dimension requirements of Theorem B can be obtained via similar means in those higher dimensional settings remains open.

Question 1. Does Theorem B hold in higher dimensions?

The $\utilde{\mathbf{\Pi}}^1_1$-Recursion Theorem 3.14 produces a $\utilde{\mathbf{\Pi}}^1_1$ set of self-constructible reals satisfying the recursion. It is well-known that the set of self-constructibles is the largest thin Π^1_1 set: it contains no perfect subset [25, 37]. As has been pointed out by Vidnyánszky [58, Problem 5.8], whether non-thin sets can solve the recursion in the $\utilde{\mathbf{\Pi}}^1_1$-Recursion Theorem remains open.

Question 2. Does there exist a $\utilde{\mathbf{\Pi}}^1_1$ set failing Marstrand's Projection Theorem which also contains a perfect subset, under suitable set-theoretic assumptions?

Secondly, we considered the property MP from the viewpoint of Hausdorff dimension. There also exists a characterisation of **packing dimension** in terms of Kolmogorov complexity, which is due to J. Lutz and N. Lutz [33, Theorem 2].

Theorem 6.1. *Let $n < \omega$ and $E \subseteq \mathbb{R}^n$. Then*

$$\dim_P(E) = \min_{A\in 2^\omega} \sup_{x\in E} \operatorname{Dim}^A(x)$$

where Dim *denotes the upper effective dimension:*

$$\operatorname{Dim}(x) = \limsup_{r\to\infty} \frac{K(\overline{x}[r])}{r}.$$

There exist bounds on the projection of subsets under $\dim_P$. However, these are less well-behaved; the best possible lower bound for $\boldsymbol{\Sigma}^1_1$ sets was isolated by J. Howroyd and K. Falconer [16], improving on M. Järvenpää's result [21].

Question 3. What packing dimensions can be realised in projections of sets of reals?

In this paper, we have constructed sets of Hausdorff dimension greater than or equal to 1. We leave open whether a $\boldsymbol{\Pi}^1_1$ set of Hausdorff dimension $\epsilon \in (0,1)$ can be constructed which also fails MP.

Question 4. Is there a $\boldsymbol{\Pi}^1_1$ set $E \subset \mathbb{R}^2$ and $\epsilon \in (0,1)$ for which $\dim_H(E) = \epsilon$ while $\dim_H(\mathrm{proj}_\theta(E)) = 0$ for all angles θ?

Finally, on the complexity side, a possible strengthening of Theorem B ensures that the witnessing set is not only $\boldsymbol{\Pi}^1_1$ but even **Borel or Wadge complete** [26, II.22.B and III.26].

Question 5. Can MP be failed by a Borel (or even Wadge) complete $\boldsymbol{\Pi}^1_1$ set?

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