Chains and Antichains inside Many-One Degrees and Variants

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Abstract

The relations between many-one degrees and one-one degrees have been studied since the beginning of recursion theory; early results from the 1960s include that many-one degrees always have a largest one-one degree and that one-one degree is either the only one-one degree inside the many-one degree or every countable linear order is noneffectively embeddable into the structure of one-one degrees inside the given many-one degree. Furthermore, the greatest recursive many-one degree is a special case, as it allows to embed ascending infinite chains but not descending infinite chains, all other many-one degrees fall into the two cases mentioned above. It remained open whether infinite antichains can always be embedded when the many-one degree is nonrecursive and infinite; Odifreddi stated in his survey from the year 1981 and in his book Classical Recursion Theory in the year 1989 this question explicitly as an open problem. Dëgtev had already in 1976 constructed antichains of one-one degrees inside all nonrecursive and nonirreducible recursively enumerable many-one degrees and Batyrshin generalised the result to all nonrecursive and nonirreducible limit-recursive many-one degrees. The present work generalises Batyrshin's result to all nonrecursive and nonirreducible many-one degrees and solves therefore Odifreddi's open problem.

The present work first proposes also to consider reducibilities between one-one and many-one, namely to study in more detail than before the finite-one and bounded finite-one degrees. Odifreddi's Open Problem is solved by showing that every nonrecursive finite-one degree which does not coincide with the greatest one-one degree in a many-one degree contains an infinite antichain of one-one degrees and furthermore allows to embed any recursive partial order effectively into the structure of one-one degrees inside the finite-one degree. This is done by starting with a representative A of the finite-one degree and then constructing an array $B_0, B_1, B_2, \ldots$ of sets given by finite-one reductions to A which are also all one-one above A and which form an antichain or embed a given recursive partial order. In contrast to this, there are nonrecursive bounded finite-one degrees consisting of a linearly ordered set of one-one degrees without any incomparable pair of one-one degrees inside it. Furthermore, some initial results about the structure of finite-one degrees inside many-one degrees are obtained, some of those using relativisations of finite-one equivalence to oracles.

Keywords and phrases Structures inside degrees; one-one degree; finite-one degree; bounded finite-one degree; many-one degree; infinite antichains.

1 Introduction

Post [23] investigated in his paper the classical reducibilities many-one, truth-table and Turing in order to determine which of them had intermediate recursively enumerable degrees besides the recursive degree and the degree of the halting problem; he answered it positively for all strong degrees, but left it open for Turing reducibility. The strongest form of reducibility are the many-one reducibility and their variants. Here f reduces A to B iff for all x, $x \in A \Leftrightarrow f(x) \in B$; the variants satisfy the additional request that f is one-one or finite-one. In all cases, for recursion theory, f has to be a recursive function. The work initiated by Post led to a comprehensive body of research comparing and relating strong reducibilities and their degrees. Odifreddi surveyed in an article [20] and in his books "Classical Recursion Theory" [21, 22] also the structure of one-one degrees inside many-one degrees; the first book concentrated on general degrees while the second book specialised at limit-recursive and recursively enumerable degrees.

Young [31] showed in 1966 that there are two base-cases for many-one degrees, either they consist of a single one-one degree or they consist of infinitely many one-one degrees. For example, the many-one degree of the empty set or of the full set of natural numbers $\mathbb{N}$ consist of a single set and thus a single one-one degree. Myhill [17] had shown already in 1955 that the many-one degree of the halting problem consists of sets which are pairwise equivalent by a recursive bijection [17], thus the many-one degree of the halting problem is a single one-one degree; such many-one degrees are called irreducible. On the other hand, simple sets as introduced by Post [23] satisfy that their many-one degree consists of infinitely many one-one degrees. Young [31] furthermore showed that the recursive and nonrecursive nonirreducible many-one degrees differ. While the greatest recursive many-one degree allows only to embed linear orders isomorphic to either a finite ordering or to the natural numbers with their default ordering or to the natural numbers with their default ordering plus one element above them, every nonrecursive nonirreducible many-one degree allows to embed any countable linear order into the structure of one-one degrees which it contains. In particular the dense linear order is embeddable into the one-one degrees inside any given nonirreducible nonrecursive many-one degree. Young worked also on other aspects of many-one and one-one degrees [29, 30].

Rogers [24] showed that every many-one degree contains a greatest one-one degree and this one-one degrees consists of all the cylinders in the many-one degree. Here, a cylinder is a set A which is one-one equivalent to the Cartesian product of A with the set of the natural numbers. Dekker and Myhill [6] showed that there are many-one degrees without a least one-one degree, examples of these are the many-one degrees of simple sets. Furthermore, the greatest recursive many-one degree has two minimal one-one degrees, the singleton sets and their complements; this degree indeed contains antichains of length 2 but not of length 3. One of the corollaries to the results in the present work is that this degree is also the only many-one degree with this property. Motivated by Young's result on the embeddability of countable linear orders, which did not include an embeddability result for countable partial orders, Dëgtev [7] showed that recursively enumerable nonrecursive and nonirreducible many-one degrees can embed countable partial orders including infinite antichains into the structure of one-one degrees inside them.

This motivated Odifreddi [20, Open Problem 5] to ask explicitly whether every nonrecursive and nonirreducible manyone degree contains an infinite antichain of one-one degrees. Batyrshin [1] confirmed this for limit-recursive many-one degrees. Here a limit-recursive set is a set Turing reducible to the halting problem K. The main result of this paper is to generalise this result to all many-one degrees, thus answering Odifreddi's question from

1981 affirmatively.

Furthermore, the present work analyses the situation when taking the intermediate finite-one degrees between the one-one degrees and many-one degrees into account. A published reference to the use of finite-one reducibility is by Kjos-Hanssen and Webb [13]; they used it as a tool to study various forms of randomness. Finite-one reducibility is the most natural candidate to sit between one-one and many-one reducibility.

The following is now known about finite-one degrees inside many-one degrees: A finite-one degree containing the greatest finite-one degree inside its many-one degree is irreducible and consists exactly of this one-one degree. All other nonrecursive finite-one degrees contain infinite antichains of bounded finite-one degrees. Those in turn might or might not contain antichains of one-one degrees, there are infinitely many bounded finite-one degrees whose one-one degrees inside them are order-isomorphic to the natural numbers with their default order. The collection of recursive sets consists of five finite-one degrees which all consist of a single bounded finite-one degree and out of which three are a single one-one degree and two are ascending chains of one-one degrees.

The interested reader finds, besides the information provided in the textbooks of Odifreddi [21, 22] and his survey article [20], also further background on strong reducibilities and recursion theory in general in other recursion-theoretic textbooks like those of Calude [3], Chong and Yu [5], Downey and Hirschfeldt [11], Li and Vitányi [18], Nies [19], Rogers [24] and Soare [27].

2 One-one degrees inside finite-one degrees

The three main reducibilities studied in the present work are many-one, one-one and finite-one; these type of reducibilities had been used by Post [23], Myhill [17] and Kjos-Hanssen and Webb [13], respectively. Reducibilities are a core concept of recursion theory and Post [23] studied them in order to get initial results towards his question whether there are recursively enumerable Turing degrees which are neither recursive nor complete (= Turing equivalent to the halting problem). The distinction between one-one, finite-one and many-one functions was already studied for centuries in other branches of mathematics, with one-one and many-one being the most common types. Finite-one functions were explicitly used in the construction of the Rudin-Blass ordering between ultrafilters in set theory, see Laflamme and Zhu [15] for an important paper on that notion; this reducibility is not really a finite-one reducibility itself, but it uses finite-one functions between the spaces on which the ultrafilters are build as part of its definition. Sets $A, B, \ldots$ used in this paper are subsets of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and sets are identified with their characteristic functions, so if $x \in A$ then $A(x) = 1$ else $A(x) = 0$.

▶ Definition 1. A function f is a many-one reduction from A to B iff f is recursive and for all x, $A(x) = B(f(x))$. Furthermore, a many-one reduction f is a finite-one reduction if for each y there are at most finitely many x with $f(x) = y$ and a one-one reduction if for each y there is at most one x with $f(x) = y$. Between these two reductions is the bounded finite-one reduction where there is a constant c such that for every y there are at most c numbers x with $f(x) = y$. Furthermore, let $X \oplus Y = \{2z : z \in X\} \cup \{2z + 1 : z \in Y\}$ be the join of X and Y. Now X, Y are both one-one reducible to their join. The many-one degree of A is the set of all B such that first B is many-one reducible to A and second A is many-one reducible to B; finite-one, bounded finite-one and one-one degrees are defined analogously.

▶ **Theorem 2.** *Let A be an infinite and coinfinite subset of the natural numbers. Then one can construct an array $B_0, B_1, \ldots$ of sets finite-one equivalent to A such that each of them is one-one above A and either the $B_1, B_2, \ldots$ forms an antichain in the one-one degrees or A is in the greatest one-one degree in its many-one degree and all B_k are one-one equivalent to A.*

Proof. One defines each set B_i via a surjective finite-one reduction h_i from natural numbers to natural numbers where $B_i(x) = A(h_i(x))$. By construction, every B_i is finite-one reducible to A and, by surjectivity, A is one-one reducible to every B_i. The functions h_i are constructed in stages and implicitly define B_i as indicated above; the main constraint is that each y has at least one and at most finitely many preimages x. The overall quantity and positions of these preimages is controlled by a finite injury construction. Furthermore, one let $\varphi_1, \varphi_2, \ldots$ be a numbering of all partial-recursive functions which are one-one on their domain and which satisfy that whenever $\varphi_e(x)$ is undefined so is $\varphi_e(x+1)$. Note that for each one-one reduction between two sets, one can find a φ_e in this list which coincides with this one-one reduction. Now one defines the following requirements $R_{e,i,j}$:

1. $R_{0,0,0}$: This requirement makes sure that for each y, i there is an x such that $h_i(x) = y$ and that each $h_i(x)$ is eventually defined.
2. $R_{e,i,j}$ with $e > 0$ and $i \neq j$: This requirement makes sure that whenever φ_e is a one-one reduction from B_i to B_j then a finite variant of a strictly increasing selfreduction from A to A is constructed - thus such a selfreduction exists and A is a cylinder, that is, the finite-one degree of A consists of a single one-one degree which is the largest one-one degree in the many-one degree of A.

For this, one assumes a default enumeration of all requirements with $R_{0,0,0}$ coming first and $R_{e,i,j}$ does not come strictly before $R_{e',i',j'}$ when $e' \leq e \wedge i' \leq i \wedge j' \leq j$. Let $g(R_{e,i,j})$ be the natural number assigned to the requirement $R_{e,i,j}$ and assume that g is a recursive bijection. In other word, if $d = g(R_{e,i,j})$ then $R_{e,i,j}$ is the d-th requirement with $R_{0,0,0}$ being the zeroth requirement.

▶ Algorithm 3. The algorithm to construct the array runs in stages. Each stage takes so long until all the requirements in it have done the actions linked to it or are skipped due to not requiring any current action.

At stage s, one first satisfies the requirement $R_{0,0,0}$:

While there are $i, y \leq s$ such that there is no x on which $h_i(x)$ is already defined to be y then take the first x where $h_i(x)$ is not yet defined and let $h_i(x) = y$. Furthermore, while there are $x, i \leq s$ with $h_i(x)$ being undefined, let $h_i(x) = y$ for the first y not yet in the range of h_i.

Now, one looks for $d = 1, 2, \ldots, s$ at that requirement $R_{e,i,j}$ which satisfies $g(R_{e,i,j}) = d$:

If the marker $m_{e,i,j}$ is currently not sitting on some number then do the following: Define the set $D = \{y : d \leq y \leq s$ and $f_d(y)$ is not already defined and none of the markers for requirements $R_{e',i',j'}$ with $0 < g(R_{e',i',j'}) < d$ is sitting on $y\}$; if D is not empty then place $m_{e,i,j}$ on the minimum of D and remove all other markers sitting there (they have lower priority).

If now the marker is sitting on a number y and $f_d(y)$ is not yet defined then one does the following steps.

While the number of x with $h_i(x) = y$ is less or equal to the number of x' with $h_j(x') \leq y$ do begin select the least x where $h_i(x)$ is not yet defined and let $h_i(x) = y$ end.

If $\varphi_e(x)$ is defined on all x where currently $h_i(x) = y$ within s steps then one finds an x with $h_i(x) = y$ and either $h_j(\varphi_e(x))$ is not yet defined or $h_j(\varphi_e(x)) > y$; fix this x for now. If $h_j(\varphi_e(x))$ is not yet defined then one defines $h_j(\varphi_e(x)) = y'$ for the first $y' > y$ which is not yet in the range of h_j, so that from now on the second subcase $h_j(\varphi_e(x)) > y$ holds. Now one defines that $f_d(y) = h_j(\varphi_e(x))$.

Now one concludes the step by removing the marker $m_{e,i,j}$ from its position if $f_d(y)$ has been defined. The marker remains on its position if the marker is waiting for future stages until all $\varphi_e(x)$ with $h_i(x) = y$ become defined.

End of activity for $R_{e,i,j}$ inside stage s.

Once all requirements $R_{e,i,j}$ with $g(R_{e,i,j}) \leq d$ are handled, this is the end of stage s and the algorithm goes to stage $s + 1$.

Recall that $B_i(x) = A(h_i(x))$ for all i, x. Thus every B_i is many-one equivalent to A via h_i. Theorem 2 will now be proven by a series of four claims showing important properties of the array of sets constructed with Algorithm 3; note that A is a parameter, but it is only evaluated at one point: To make a finite modification of some function f_d to show that A is a cylinder in the case that the B_i do not form an antichain of one-one degrees inside the finite-one degree of A.

▶ Claim 4. At every stage, all h_i have a finite domain and overall only finitely many new values of the functions f_i are defined in a stage.

To see this claim, assume by way of contradiction that this would be false and let s be the first stage where for infinitely many pairs a value $h_i(x)$ is newly defined. Furthermore, there must be a first requirement with respect to the requirement number $d = g(R_{e,i,j})$ for which this happens. It cannot be that $d = 0$ as that requirement defines at most for each (i, y) with $i, y \leq s$ one value $h_i(x) = y$ and furthermore for each (i, x) with $i, x \leq s$ at most one further value. This are at most $2 \cdot (s+1)^2$ many definitions and thus only finitely many. For $d > 0$, there are only two definition steps inside the activity. The substep

> "While the number of x with $h_i(x) = y$ is less or equal to the number of x' with $h_j(x') \leq y$ do begin select the least x where $h_i(x)$ is not yet defined and let $h_i(x) = y$ end."

defines only finitely many values $h_i(x)$ as by assumption there are only finitely many value $h_j(x)$ defined before. Furthermore, there is in this step at most one further point where a value is defined and that is defining $h_j(x) = y'$ for some value $y' > s$. Thus in contrary to the assumption, only finitely many new definitions are done in this step. Therefore also at most finitely many new definitions are done in stage s.

▶ Claim 5. For all i, y there are at least one and at most finitely many x with $h_i(x) = y$. In particular each B_i is finite-one equivalent to A and A is one-one reducible to each B_i.

Note that $R_{0,0,0}$ enforces that for each pair (i, y) there is at least one x with $h_i(x) = y$ and that therefore all functions h_i are surjective. Thus one has only to show that the overall number is finite. By the first verification item, this can only happen if some marker sits forever on a value y for a requirement $R_{e,i,j}$. So let y be the least number so that there are infinitely many x with $h_i(x) = y$ for one single i. Furthermore, this can only be caused by finitely many requirements, as requirements $R_{e,i,j}$ with $g(R_{e,i,j}) > y$ do not define h_i on any x to have the value y. In addition all the requirements together who act at markers strictly below y can only define $h_i(x)$ for the first x which takes the value y but not for further ones, thus these markers cannot cause the problem. So it must be one requirement $R_{e,i,j}$ whose marker settles on y forever. Let s be so large that all requirements which define some value $h_j(x') = y'$ for some $y' < y$ have already done so, by choice of y there are only finitely many; furthermore, s is so large such that there is at least one x with $h_j(x) = 0$ already being defined. Furthermore the marker $m_{e,i,j}$ sits from stage s onwards forever on y, and the markers of the higher priority requirements are either sitting on values strictly above y

forever or they have converged to a lower value and will not be moved again. Thus, no other marker makes new definitions of the form $h_j(x') = y'$ for some $y' \leq y$ from now on. Let t be the number of x with $h_j(x) \leq y$, this number is thus constant. It follows by the way that the activity related to requirement $R_{e,i,j}$ is defined that there are at most $t+1$ many values x for which the requirement defines $h_i(x) = y$, thus there are only finitely many and not infinitely many. So the assumption was false and it follows from contraposition that each B_i is indeed finite-one equivalent to A via the mapping h_i.

▶ Claim 6. Assume that φ_e one-one reduces B_i to B_j and $d = g(R_{e,i,j})$. Now f_d is defined on almost all inputs and for all y in its domain, f_d satisfies $A(y) = A(f_d(y))$ and $f_d(y) > y$.

To see this claim, consider any y in the domain of f_d. Then $f_d(y) = h_j(\varphi_e(x))$ for some x with $h_i(x) = y$ and $h_j(\varphi_e(x)) > y$. Thus $f_d(y) > y$. Furthermore, $B_i(x) = A(h_i(x))$ by definition and $B_j(\varphi_e(x)) = A(y)$ by the assumed correctness of φ_e and the equality-chain $A(f_d(y)) = A(h_j(\varphi_e(x))) = B_j(\varphi_e(x)) = B_i(x) = A(y)$ shows that A is a partial selfreduction on A.

Now assume that $y \notin dom(f_d)$ and $y \geq d$. There are only finitely many values on which a marker of a higher priority requirement stays forever, so assume that y is not one of them. One possibility is now that the marker $m_{e,i,j}$ itself stays forever on y. Then there is some stage s large enough such that all x with $h_i(x) \leq y$ or $h_j(x) \leq y$ are already defined and $\varphi_e(x)$ is defined for all x with $h_i(x) = y$ and $m_{e,i,j}$ is sitting on y. Then it must be that there are more x with $h_i(x) = y$ than x with $h_j(x) \leq y$, otherwise $h_i(x) = y$ would become defined for some further x in the future; as this does not happen, this must already have been defined before. Now there must be an x such that $h_i(x) = y$ and $h_j(\varphi_e(x))$ is either undefined or strictly above y; however, this would force in this stage the value of $f_d(y)$ to become defined. As by assumption this does not happen, it cannot be that the marker $m_{e,i,j}$ sits on some y forever. Furthermore, it cannot be that the y is overlooked, that is, from some time point onwards, for all further stages, none of the markers associated to a requirement $R_{e',i',j'}$ with $g(R_{e',i',j'}) \leq d$ sits on y. Without loss of generality, y is the least one of the numbers greater equal to d for which $f_d(y)$ is undefined and to which none of the markers $m_{e',i',j'}$ with $g(R_{e',i',j'}) \leq d$ converges. Then y would for almost all stages qualify as the value on which $m_{e,i,j}$ will take and thus, in contrary to the assumption, $f_d(y)$ gets defined. This completes the proof that the domain of f_d is cofinite.

▶ Claim 7. If φ_e is a one-one reduction from B_i to B_j then A is a cylinder, that is, in the greatest one-one degree of its many-one degree. Furthermore, if A is infinite, coinfinite and not a cylinder, then $B_0, B_1, \ldots$ is an antichain inside the finite-one degree of A.

By the above, f_d is defined for almost all y. One makes it total by mapping all remaining y to the first $y' > y$ with $A(y') = A(y)$. Now let $f_{d,0}(x) = x$ and $f_{d,k+1}(x) = f_d(f_{d,k}(x))$ for all k. Using the functions $f_{d,k}$ one can, given a many-one reduction f' from some set C to A, one can also obtain a one-one reduction f'' by defining for $x = 0, 1, \ldots$ the value $f''(x) = f_{d,k}(f'(x))$ for the first k where $f_{d,k}(f'(x)) \notin \{f''(x') : x' < x\}$. This proves that C is one-one reducible to A and thus A is in the greatest one-one degree of its many-one degree.

If now A is infinite and coinfinite and not a cylinder, then no f_d can be finitely extended to a total strictly increasing selfreduction of A, thus either f_d is not a partial selfreduction or its domain is coinfinite. This happens only if the corresponding φ_e is not a one-one reduction from B_i to B_j. Thus for the array of the B_i constructed, there exist no distinct indices i, j and no one-one reduction φ_e such that φ_e reduces B_i to B_j. So $B_0, B_1, \ldots$ is an antichain in the one-one degrees, the property that all B_i are strictly one-one above A but still finite-one equivalent to A follows from the construction. ◀

Theorem 2 can be improved to the following Theorem 8. For this, let $\sqsubset$ be a recursive partial order, that is, a relation which is transitive and which satisfies for all distinct i, j that either (a) $i \sqsubset j$ or (b) $j \sqsubset i$ or (c) i, j are incomparable; note that the case $i \sqsubset j \wedge j \sqsubset i$ does not occur. An algorithm can compute for each pair of distinct i, j which of the above three cases (a), (b) and (c) applies.

▶ **Theorem 8.** *Let $\sqsubset$ be any recursive partial order. Let A be an infinite and coinfinite set which is not in the greatest one-one degree of its many-one degree. Then one can construct $B_0, B_1, \ldots$ finite-one equivalent to A such that B_i is one-one reducible to B_j if and only if either $i = j$ or $i \sqsubset j$; that is, one can embed every recursive partial order into the one-one degrees inside the finite-one degree of A.*

Proof. The proof of Theorem 8 is similar to that of Theorem 2. Thus it is mainly listed out what changes are to be done to prove the result. Let $\sqsubseteq$ be a recursive preorder, that is, it is transitive and reflexive. Again one defines each set B_i via a surjective finite-one reduction h_i from natural numbers to natural numbers where $B_i(x) = A(h_i(x))$. By construction, every B_i is finite-one reducible to A and, by surjectivity, A is one-one reducible to every B_i. The functions h_i are constructed in stages and implicitly define B_i as indicated above; the main constraint is that each y has at least one and at most finitely many preimages x. The overall quantity and positions of these preimages is controlled by a finite injury construction. Furthermore, one let $\varphi_1, \varphi_2, \ldots$ be an acceptable numbering of all partial-recursive one-one functions with $\varphi_e(x+1)$ only being defined when $\varphi_e(x)$ is and one considers the following requirements $R_{e,i,j}$:

1. $R_{0,0,0}$: This requirement makes sure that for each y, i there is an x such that $h_i(x) = y$ and that each $h_i(x)$ is eventually defined.
2. $R_{0,i,j}$ with $i \neq j$ and $j \sqsubset i$: This requirement makes sure that, for almost all y, there are at strictly more x with $h_i(x) = y$ as there are x' with $h_j(x') \leq y$. Thus, for all but finitely many y, one can map the x' with $h_j(x') = y$ in a one-one way to the x with $h_i(x) = y$ and one can use A' to map the remaining finitely many x' to counterparts x with $B_i(x) = B_j(x')$.
3. $R_{e,i,j}$ with $e > 0$ and $i \neq j$ and $i \not\sqsubset j$: This requirement makes sure that whenever φ_e is a one-one reduction from B_i to B_j then a finite variant of a strictly increasing selfreduction from A to A is constructed—thus, such a selfreduction exists and A is a cylinder, that is, the finite-one degree of A consists of a single one-one degree which is the largest one-one degree in the many-one degree of A.

For this, one assumes a default enumeration of all requirements with $R_{0,0,0}$ coming first and $R_{e,i,j}$ does not come strictly before $R_{e',i',j'}$ when $e' \leq e \wedge i' \leq i \wedge j' \leq j$. Let $g(R_{e,i,j})$ be the natural number assigned to the requirement $R_{e,i,j}$ and assume that g is a recursive bijection.

▶ Algorithm 9. The algorithm runs in stages. Each stage takes so long until all the requirements in it have done the actions linked to it.

At stage s, one first satisfies the requirement $R_{0,0,0}$:

While there are $i, y \leq s$ such that there is no x on which $h_i(x)$ is already defined to be y then take the first x where $h_i(x)$ is not yet defined and let $h_i(x) = y$. Furthermore, while there are $x, i \leq s$ with $h_i(x)$ being undefined, let $h_i(x) = y$ for the first y not yet in the range of h_i.

Now, one looks for $d = 1, 2, \ldots, s$ at the requirement $R_{e,i,j}$ with $g(R_{e,i,j}) = d$:

If the marker $m_{e,i,j}$ is currently not sitting on some number then do the following: Define the set $D = \{y : d \leq y \leq s$ and (either $f_d(y)$ is not already defined or $e = 0$ and there are at least as many x' with $h_j(x') \leq y$ as x with $h_i(x) = y$) and none of the markers for requirements $R_{e',i',j'}$ with $0 < g(R_{e',i',j'}) < d$ is sitting on $y\}$; if D is not empty then place

$m_{e,i,j}$ on the minimum of D and, if $e > 0$, then remove all other markers sitting there (they have lower priority).

If now the marker $m_{e,i,j}$ is sitting on a number y then one does the following steps.

While the number of x with $h_i(x) = y$ is less or equal to the number of x' with $h_j(x') \leq y$ do begin select the least x where $h_i(x)$ is not yet defined and let $h_i(x) = y$ end.

If $e > 0$ and $\varphi_e(x)$ is defined on all x where currently $h_i(x) = y$ within s steps then one finds an x with $h_i(x) = y$ and either $h_j(\varphi_e(x))$ is not yet defined or $h_j(\varphi_e(x)) > y$; fix this x for now.

If $h_j(\varphi_e(x))$ is not yet defined then one defines $h_j(\varphi_e(x)) = y'$ for the first $y' > y$ which is not yet in the range of h_j, so that from now on the second subcase $h_j(\varphi_e(x)) > y$ holds. Now one defines that $f_d(y) = h_j(\varphi_e(x))$.

Now one concludes the step by removing the marker $m_{e,i,j}$ from its position if either $e = 0$ or $f_d(y)$ has been defined. It remains on its position if (a) the marker is waiting for future stages until all $\varphi_e(x)$ with $h_i(x) = y$ become defined and (b) no higher priority marker with $e > 0$ goes onto y.

End of activity for $R_{e,i,j}$ inside stage s.

Once all requirements $R_{e,i,j}$ with $g(R_{e,i,j}) \leq d$ are handled, this is the end of stage s and the algorithm goes to stage $s + 1$.

Recall that $B_i(x) = A(h_i(x))$ for all i, x. Thus every B_i is many-one equivalent to A via h_i. Again, one will establish the properties of the array of sets constructed by above algorithm 9 through a series of four claims.

▶ Claim 10. At every stage, all h_i have a finite domain and overall only finitely many new values of the functions f_i are defined in a stage.

Assume by way of contradiction that this claim would be false and let s be the first stage where for infinitely many pairs a value $h_i(x)$ is newly defined. Furthermore, there must be a first requirement with respect to the requirement number $d = g(R_{e,i,j})$ for which this happens. It cannot be that $d = 0$ as that requirement defines at most for each (i, y) with $i, y \leq s$ one value $h_i(x) = y$ and furthermore for each (i, x) with $i, x \leq s$ at most one further value. This are at most $2 \cdot (s + 1)^2$ many definitions and thus only finitely many. For $d > 0$, there are only two definition steps inside the activity. The substep

> "While the number of x with $h_i(x) = y$ is less or equal to the number of x' with $h_j(x') \leq y$ do begin select the least x where $h_i(x)$ is not yet defined and let $h_i(x) = y$ end."

defines only finitely many values $h_i(x)$ as by assumption there are only finitely many value $h_j(x)$ defined before. Furthermore, there is in this step at most one further point where a value is defined and that is defining $h_j(x) = y'$ for some value $y' > y$. Thus in contrary to the assumption, only finitely many new definitions are done in this step. Therefore also at most finitely many new definitions are done in stage s.

▶ Claim 11. For all i, y there are at least one and at most finitely many x with $h_i(x) = y$. In particular each B_i is finite-one equivalent to A and A is one-one reducible to each B_i.

Note that $R_{0,0,0}$ enforces that for each pair (i, y) there is at least one x with $h_i(x) = y$. Thus one has only to show that the overall number is finite.

So assume that there are pairs (i, y) with infinitely many x satisfying $h_i(x) = y$. Among those pairs, take y as small as possible and fix it from now on.

There are at most y indices i for which there are at least two x with $h_i(x) = y$. The reason is that for each such i there must be a requirement $R_{e,i,j}$ where $e = 0$ is possible with $j \neq i$ and $1 \leq g(R_{e,i,j}) \leq y$. Note that except for the first x with $h_i(x) = y$, all further ones are defined by some requirement $R_{e,i,j}$ with $g(R_{e,i,j}) \leq y$ and i must be the second parameter, not the third of the requirement.

Now let $E = \{i, j$: there exists requirement $R_{e,i,j}$ with $g(R_{e,i,j}) \leq y$ and $j \neq i\}$, let $E' = \{i \in E$: there are infinitely many x with $h_i(x) = y\}$ and let $E'' = \{i \in E'$: no $j \sqsubset i$ is in $E'\}$. Note that E, E', E'' are finite sets and as $\sqsubset$ is a partial order, $E' \neq \emptyset$ implies $E'' \neq \emptyset$ and that only the $i \in E$ satisfy that there are for some $y' \leq y$ at least two x' with $h_i(x') = y'$. For each $i \in E''$ there must be be a marker $R_{e,i,j}$ with $g(R_{e,i,j}) \leq y$ sitting on y infinitely often in order to achieve that $i \in E'$, that is, that there are infinitely many x with $h_i(x) = y$. Therefore one has to look at the stages and marker movement in more detail.

So let s be so large that the following holds:

1. $s \geq y$.
2. If $i \in E$ and $f_d(y)$ gets eventually defined then this happened before stage s.
3. All requirements which define only finitely many $h_i(x)$ with $h_i(x) \leq y$ and $i \in E$ have done this before stage s.
4. All markers which go only finitely often onto a number $y' \leq y$ have completed these actions before stage s.
5. All requirements $R_{e,i,j}$ with $g(R_{e,i,j}) \leq y$ which have a marker only finitely often sitting on some $y' \leq y$ have removed this marker forever from the corresponding y' before stage s.

Now let $F = \{$Requirement $R_{e',i',j'}$: $m_{e',i',j'}$ is on y for infinitely many stages and $e' > 0\}$ and let $R_{e,i,j}$ be the member of F where $d = g(R_{e,i,j})$ is minimal. Then $f_d(e)$ must remain undefined forever and therefore the marker will not be released; the only higher priority markers which may get attention are those where $e = 0$ and those do not remove $m_{e,i,j}$ from its current position; note that by conditions 2 and 3 above in the choice of s, markers which get attention only finitely often do this before stage s and will not be requesting it again at stage s or later. Therefore the only way to assign a new $h_{i'}(x') = y$ for an $i' \in E$ is when either $i' = i$ and the number of x'' with $h_j(x'') \leq y$ has increased before or when there is a $i'' \sqsubset i'$ where $i'' \in E$ and the number of x'' with $h_{i''}(x'') \leq y$ has increased after stage s. However, this requires that $i = i'$ or $i \sqsubset i'$ by the fact that $m_{e,i,j}$ is sitting on y and does not make space for other markers. Furthermore, $j \neq i$ and $j \not\sqsubset i$, thus there are no new x' with $h_{i'}(x) = y$ and $i' = j \vee i' \sqsubset j \vee i' \sqsubset i$. Therefore no new x with $h_i(x) = i$ are added and, in contrary to the assumption, $i \notin E'$. Thus $E' = \emptyset$, that is, there is no i with $h_i(x) = y$ for infinitely many x. It follows that the stament of Claim 11 is correct.

▶ Claim 12. Assume that $e > 0$ and $i \neq j$ and φ_e is total and one-one reduces B_i to B_j and $i \not\sqsubset j$. Now the requirement $R_{e,i,j}$ exists and has some value d and f_d is defined on almost all inputs and for all y in its domain, f_d satisfies $A(y) = A(f_d(y))$ and $f_d(y) > y$.

This claim has the same proof as Claim 6.

▶ Claim 13. If the two properties φ_e is a one-one reduction from B_i to B_j and $i \not\sqsubseteq j$ jointly hold then A is a cylinder, that is, in the greatest one-one degree of its many-one degree. Furthermore, if $i \sqsubseteq j$ then B_i is one-one reducible to B_j, independent on what set A is, only provided that A is infinite and coinfinite. Thus, if A is neither recursive nor a cylinder, then $B_0, B_1, \ldots$ represent an array of one-one degrees inside the finite-one degree of A, whose ordering by one-one reducibility coincides with the partial order $\sqsubseteq$ when made reflexive by using $\sqsubseteq$ instead of $\sqsubset$ itself.

By the above, if $i \not\sqsubseteq j$ and φ_e is total then f_d is defined for almost all y. One makes it total by mapping all remaining y to the first $y' > y$ with $A(y') = A(y)$. Now let $f_{d,0}(x) = x$ and $f_{d,k+1}(x) = f_d(f_{d,k}(x))$ for all k. Using the functions $f_{d,k}$ one can, given a many-one reduction f' from some set C to A, one can also obtain a one-one reduction f'' by defining for $x = 0, 1, \ldots$ the value $f''(x) = f_{d,k}(f'(x))$ for the first k where $f_{d,k}(f'(x)) \notin \{f''(x') : x' < x\}$. This proves that C is one-one reducible to A and thus A is in the greatest one-one degree of its many-one degree.

If now A is infinite and coinfinite and not a cylinder and $d = g(R_{e,i,j})$ for a requirement with $e > 0$, then f_d cannot be finitely extended to a total strictly increasing selfreduction of A, thus either f_d is not a partial selfreduction or its domain is coinfinite. This happens only if the corresponding φ_e is not a one-one reduction from B_i to B_j. As e was chosen arbitrarily, B_i is not one-one reducible to B_j.

Furthermore, if $j \sqsubset i$, then $j \neq i$ and for all d and all $y \geq d$ with $d = g(R_{0,i,j})$, it holds that either some higher priority requirement with a number strictly below $g(0, i, j)$ has a marker eventually sitting forever on y or that there are strictly more x with $h_i(x) = y$ than x' with $h_j(x') = y$. Thus one can map the x with $h_i(x) = y$ for almost all y in a one-one way the x' with $h_j(x') = y$ to x with $h_i(x) = y$, let f be the so far constructed mapping. Now the remaining finitely many undefined places of f can be patched, as for almost all y there is an x with $h_i(x) = y$ not in the range of the f constructed so far and while there are only finitely many x' not yet mapped to an x; thus B_j is one-one reducible to B_i. The just mentioned patching can be done using the oracle A' and as that oracle is used only finitely often, a nonuniformly obtained finite table can replace it. If $j = i$ then the identity one-one reduces B_j to B_i, thus all cases of $j \sqsubseteq i$ are covered. ◀

Mostowski [16] has proven that there is a universal recursive partial order, that is, a recursive partial order $\sqsubset$ such that every further countable partial order can, though not effectively, be embedded into it. Thus one has the below corollary, where the first part follows directly from Theorem 8 and the second part is a direct consequence of the fact that when A has a nonirreducible finite-one degrees then A has also a nonirreducible many-one degree. Gu [10, Lemma 3.4] provides an outline for this, quite short, construction. Sacks [26] proved the related result that every countable partial order can be embedded into the structure of all Turing degrees with Turing reducibility as partial order.

▶ **Corollary 14.** *Let A be a nonrecursive set which is not isomorphic to a cylinder. Now every at most countable partial order $(P, \sqsubset)$ can be embedded, in a noneffective way, into the following structures:*
(a) *the partially ordered set of one-one degrees inside the finite-one degree of A;*
(b) *the partially ordered set of one-one degrees inside the many-one degree of A.*
Furthermore, every nonrecursive nonirreducible many-one (finite-one) degree has a representative A which is neither recursive nor a cylinder, thus every countable partial order can be embedded into the structure of one-one degrees inside such a degree.

▶ Remark 15. This corollary has a direct consequence: A nonirreducible many-one degree must consist of several finite-one degrees, for example the greatest recursive many-one degree consists of three finite-one degrees. The irreducible many-one degrees consist, in contrast, just of one finite-one degrees. Odifreddi [20, Problem 4] asks for providing explicit criteria of either a many-one degree or of its representatives such that the many-one degree is irreducible, these criteria should either for the structure of the degree or for one or all representing sets. An example for such a criterion is that all sets in the many-one degree are cylinders; this follows from Myhill's Isomorphism Theorem [17] which states that if two sets are in the same one-one degree then they are equivalent by a recursive permutation and therefore, if one of them is of the form $A \times \mathbb{N}$ then the other one can also be viewed as a set of pairs with an adjusted pairing function – the adjustment stems from the bijection. So for Odifreddi's Problem 4, one of the criteria that a many-one degree is irreducible is that it coincides with a finite-one degree. Another criterion is that the one-one degrees inside a given many-one degree form a chain—this chain has then to collapse to just one single one-one degree. However, the absence of antichains of length three is then only equivalent to the following property: either the many-one degree is irreducible or it coincides with the greatest recursive many-one degree. This property is then also equivalent to the statement that every finite-one degree inside the many-one degree coincides with a bounded finite-one degree, see the sections below for more information about bounded finite-one degrees.

3 Finite-one degrees inside many-one degrees

The knowledge of the structure of finite-one degrees inside a many-one degree is a bit limited compared what one knows about the structure of one-one degrees inside a many-one degree. However, there are some differences one can easily see when comparing the structure of one-one degrees inside finite-one degrees with the structure of finite-one degrees inside many-one degrees. The next proposition summarises facts about finite-one degrees, which can be easily proven with general knowledge about recursion theory. For these, $c(x, y)$ is Cantor's pairing function given as $c(x, y) = (x + y) \cdot (x + y + 1) + y$.

▶ **Theorem 16.** *The following properties hold for the structure of finite-one degrees inside many-one degrees.*

1. *The join $A \oplus B$ of two sets A, B is the least upper bound of A and B in the finite-one degrees and thus the finite-one degrees form an upper semilattice; the same applies to the structure of finite-one degrees inside many-one degrees.*
2. *The greatest finite-one degree inside a many-one degree is always irreducible, that is, coincides with a one-one degree and consists only of cylinders.*
3. *Every maximal set represents a minimal degree within the nonrecursive finite-one degrees; however, due to maximal sets being simple, there are only two of the five recursive finite-one degrees below them: The finite-one degree of the cofinite sets with at least one nonelement and the finite-one degree consisting of the single set $\mathbb{N}$.*
4. *Every bi-immune set A represents the least finite-one degree inside its many-one degree and that many-one degree contains at least two finite-one degrees. Furthermore, the many-one degree of A does not neither have a least one-one degree nor a minimal one-one degree and no recursive set is finite-one reducible to A.*

Proof. For the first item, consider A, B and assume that A, B are both finite-one reducible to a set C via f, g. Then for each y there are only finitely many v with $f(v) = y$ and finitely many w with $g(w) = y$. Now let $h(2v) = f(v)$ and $h(2w + 1) = g(w)$. Now, for all x, $(A \oplus B)(x) = C(h(x))$, hence h is a many-one reduction from $A \oplus B$ to C. Furthermore, there are, for each y, only finitely many x with $h(x) = y$, as the number of these x is the sum of the number of all v with $f(v) = y$ and the number of all w with $g(w) = y$. Thus h is a finite-one reduction witnessing that $A \oplus B$ is finite-one reducible to C. Furthermore, it is clear that both A, B are finite-one reducible to $A \oplus B$. Thus $A \oplus B$ represents the least upper bound of A and B in the finite-one degrees.

For the second item, just note that if $A \times \mathbb{N}$ represents the largest one-one degree inside a many-one degree and if $A \times \mathbb{N}$ is finite-one reducible to B in the same many-one degree via f then one can make f to be one-one as follows: Given f, one constructs g inductively over an enumeration of all pairs $c(x, y)$. If $f(c(x, y))$ is not yet in the range of g then one defines $g(c(x, y)) = f(c(x, y))$ else one finds the first z with $f(c(x, z))$ not in the so far defined range of g and let $g(c(x, y)) = f(c(x, z))$. The verification that this gives a one-one reduction which is correct is left to the reader.

For the third item, let A be a maximal set and assume that B is many-one reducible to A via f; B is therefore a recursively enumerable set. Either finitely many $y \notin A$ are in the range of f and B is recursive or all but finitely many of the $y \notin A$ are in the range of f and B is in the same many-one degree as A. Assume the second one, as only that case is interesting; now one constructs a finite-one reduction g from A to B. $g(y)$ is defined to be the x in that case which applies first; the first case applies only for finitely many y and they can be tabled up in the algorithm; which of the cases two or three strikes first in the remaining cases is determined by parallel search.

1. y is neither in the range of f nor in the set A and x is the smallest number not in B;
2. The number x is the first x found with $f(x) = y$;
3. y is enumerated into A and x is the y-th element to be enumerated into B by some fixed one-one enumeration of B; note that B is infinite and every infinite recursively enumerable set has a recursive one-one enumeration.

Independently of which case first holds, let $g(y)$ be the so found x in the corresponding case. The function is finite-one for the following reasons: The x in the first case can only be chosen by the finitely many y which are neither in the range of f nor in A as well as by one y with $f(x) = y$. The second case contributes for each x at most one y with $g(y) = x$, this is the y with $f(x) = y$. The third case produces also, for each x, at most one y with $g(y) = x$, as that x is the y-th element in a fixed recursive one-one enumeration of B. Thus, except for the smallest x not in B, each element in the range of g is only the image of at most two numbers.

For the fourth item, let A be a bi-immune set, that is, a set A such that neither A nor its complement has an infinite recursive subset. Furthermore, assume that A is many-one reducible to some set B via f. Then for each y, the set $\{x : f(x) = y\}$ is a recursive set which is either a subset of A or its complement; thus it is finite. Therefore f is a finite-one reduction to B. As A is bi-immune, A is not a cylinder and therefore there are at least two finite-one degrees in the many-one degree of A. Furthermore, the finite-one degree of A does not have a least one-one degree, as given A and $B = \{x : x+1 \in A\}$, the set B has a one-one degree strictly below that of A and is also bi-immune. Assume by contradiction that one could one-one reduce A to B via some recursive g, then let $x_0 = 0$ and $x_{n+1} = g(x_n) + 1$ for all n. For all n, $A(g(x_n)+1) = B(g(x_n)) = A(x_n)$ and $x_{n+1} \notin \{x_0, x_1, \ldots, x_n\}$, thus the sequence $x_0, x_1, \ldots$ would become an infinite recursive enumeration of numbers which are either all in A or all outside A, in contradiction to the bi-immunity of A. Furthermore, no recursive set is finite-one reducible to any bi-immune set, as each recursive set is infinite or coinfinite and thus, if the reduction would exist, A would have an infinite recursive subset or its complement would have an infinite recursive subset. ◀

The results obtained so far allow to classify the finite-one degrees into three groups. The first and the third group have uncountably many members each (given by cylinders and the biimmune sets which both exist uncountably often and from each group, only countably many can go into one finite-one degree) and the second group consists only of two finite-one degrees, both contained in the greatest recursive many-one dgree.

► Corollary 17. Assume that A is a set. Then for the finite-one degree of A, exactly one of the following statements is true and each possibility can occur.

1. The finite-one and one-one degree of A coincide and A is in the greatest one-one degree of its many-one degree;
2. The finite-one degree of A is an ascending chain isomorphic to the natural numbers with their natural ordering and consists either of all nonempty finite sets or of all cofinite sets with at least one nonelement;

3. Every recursive partial order (including an antichain) can be embedded into the finite-one degree of A with each representative of the antichain being strictly one-one above A itself; these finite-one degrees also do not contain a greatest one-one degree.

The collapse in the first case follows from the second item in Theorem 16. The second item is well-known and the third item follows from Theorems 2 and 8 and is explictely stated in Corrolary 14.

Note that there are five recursive finite-one degrees, the two mentioned in the second item of Corollary 17 and the degrees $\{\emptyset\}$ and $\{\mathbb{N}\}$ and the finite-one degree of all infinite and coinfinite recursive sets, the last three consist all of a single one-one degree.

4 Bounded finite-one degrees

In recursion theory, bounded reducibilities (say, bounded truth-table, bounded weak truth-table and bounded Turing) are given by algorithms (of type truth-table, weak truth-table and Turing, respectively) which reduce a set A to B in a way that for every x, $A(x)$ is determined by querying at most c values for some constant c which is independent of x; for these reducibilities, see for example the references [2, 21]. Similarly one can also define bounded finite-one reducibility as a variant of finite-one reducibility; here, however, one bounds the sizes of preimages by a constant, as there is anyway only one query per input. Thus, as defined in Definition 1, a bounded finite-one reduction f from A to B is a finite-one reduction from A to B with the additional constraint that there is a constant c such that for each y there are at most c numbers x with $f(x) = y$. Although this looks very similar to finite-one degrees and although these degrees share various properties like forming an upper semilattice with the standard join, there is a big difference: Nonrecursive bounded finite-one degrees might consist of infinite ascending chains of one-one degrees where each two of them are comparable; this differs strongly from the case of finite-one degrees where the nonrecursive degrees are either irreducible or contain antichains of one-one degrees. The following result provides the construction of such bounded finite-one degrees.

▶ **Theorem 18.** *There is a nonrecursive bounded finite-one degree such that its one-one degrees form an ascending chain of the same order type as the natural numbers.*

Proof. Let E_0 be a maximal set with complement E_3 and split E_0 using the Sacks splitting theorem [25] into two recursively enumerable sets E_1 and E_2 of incomparable Turing degree; once that is done, one can choose using the hyperimmune-free basis theorem of Jockusch and Soare [12] a hyperimmune-free set A such that A is a superset of E_1 and its complement a superset of E_2.

Now let B be a set in the bounded finite-one degree of A. One now shows that B is one-one equivalent to the join of ℓ copies of A for some natural number $\ell > 1$.

To see this, one considers a bounded finite-one reduction f from B to A. Furthermore, let ℓ be the biggest number such that for infinitely many $y \in E_3$ there are ℓ different x with $f(x) = y$; due to f being bounded finite-one, such a maximal ℓ must exist. Furthermore, $\ell > 0$ as otherwise f maps almost all numbers either to E_1 or to E_2 which allows to construct a decision-procedure for B, that is, B would be recursive and not in the bounded finite-one degree of A. This implies that almost all members of E_3 are ℓ times in the range of f and moving finitely many elements from E_3 to E_1 and E_2 as in Theorem 23 will result in all elements of the new E_3 appearing exactly ℓ times in the range of f. There are uniformly recursive ascending unions with $E_1 = \cup_s E_{1,s}$, $E_2 = \cup_s E_{2,s}$ and $E_{1,0}$ and $E_{2,0}$ being infinite. For each x, let $y = f(x)$ and k be the number such that x is the k-th number mapped to y

when looking at the x in ascending order. Furthermore, let s be the least number such that one of the following cases applies and define h by the first case which applies.

1. If $y \in E_{1,s}$ then let $h(x)$ be the first element of $E_1 \times \{1, 2, \ldots, k\}$ not yet in the range of h;
2. If $y \in E_{2,s}$ then let $h(x)$ be the first element of $E_2 \times \{1, 2, \ldots, k\}$ not yet in the range of h;
3. If $k \leq \ell$ and $y \notin E_{1,s} \cup E_{2,s}$ then let $h(x) = c(y, k)$.

By construction, h is one-one. Furthermore, the range of h is $\mathbb{N} \times \{1, 2, \ldots, \ell\}$, as all numbers $y \in E_3$ are exactly ℓ times in the range of f and all numbers $y \in E_1 \cup E_2$ are eventually enumerated into E_1 or E_2, respectively, and then mapped in a bijective way to the target set following its enumeration. Thus B is one-one equivalent to the ℓ-fold selfjoin of A. This directly implies that the one-one degrees inside the bounded finite-one degree of A are linearly ordered, as B is one-one reducible to C whenever it holds that B is one-one equivalent to the ℓ-fold selfjoin of A and C is one-one equivalent to the ℓ'-fold selfjoin of A and $\ell \leq \ell'$. It remains to show that the hierarchy stands and that the bounded finite-one degree of A is not irreducible.

Assume now by way of contradiction that h' one-one reduces the $\ell + 1$-fold selfjoin B of A to the ℓ-fold selfjoin C of A. Now define the following equivalence relation on the set of natural numbers:

Now let $\sim$ be the smallest equivalence relation which for all x, y enforces $x \sim y$ whenever one of the below conditions holds:

1. $x = y$ (enforcing reflexiveness);
2. $y \sim x$ (enforcing symmetry);
3. There are $k, z_1, z_2, \ldots, z_k$ with $x \sim z_1$ and $z_1 \sim z_2$ and $\ldots$ and $z_k \sim y$ (enforcing transitiveness);
4. x, y are both in E_1 (enforcing all of E_1 to go into one equivalence class);
5. x, y are both in E_2 (enforcing all of E_2 to go into one equivalence class);
6. There are $i \in \{1, 2, \ldots, \ell + 1\}$ and $j \in \{1, 2, \ldots, \ell\}$ with $h'(c(x, i)) = c(y, j)$.

So $\sim$ is an recursively enumerable equivalence class which respects A, that is, $x \sim y$ implies that either both x, y are in A or none of x, y is in A. The reason for this is that none of the above rules enforces that an element of A and a nonelement of A become equivalent.

By the cohesiveness of E_3, an equivalence class of $\sim$ either has a finite intersection with E_3 or contains almost all elements of E_3. The latter cannot happen as both A and its complement have an infinite intersection with E_3. Thus all equivalence classes have only a finite intersection with E_3. Therefore, only finitely many elements of E_3 belong to the equivalence classes of E_1 and E_2 and there must be further finite equivalence classes which are a complete subset of E_3. Let E_4 be such a finite equivalence class and let k be the number of its elements. Then h' maps the $k \cdot (\ell + 1)$ elements of $E_4 \times \{1, 2, \ldots, \ell + 1\}$ to the $k \cdot \ell$ elements of $E_4 \times \{1, 2, \ldots, \ell\}$. Thus h' cannot be one-one and therefore the bounded finite-one degree of A consists of infinitely many one-one degrees. ◀

▶ Remark 19. If one does the above construction of Theorem 18 with two Turing incomparable maximal sets E_4 and E_5 and then considers $A \oplus B$ for the so obtained sets A and B of hyperimmune-free degree, then the one-one degrees of the bounded finite-one degree of $A \oplus B$ consider of all $C_{i,j}$ being joins of i copies of A and j copies of B with $i, j \geq 1$. Now $C_{i,j}$ is one-one reducible to $C_{i',j'}$ if and only if $i \leq i'$ and $j \leq j'$. Thus the one-one degrees are partially ordered the same way as pairs of natural numbers and so contain finite antichains of arbitrary length but no infinite antichains. An example of a finite antichain of length

5 is $(5,1),(4,2),(3,3),(2,4),(1,5)$ which translates into the antichain $C_{5,1},C_{4,2},\ldots,C_{1,5}$ of one-one degrees inside the bounded finite-one degree of $A \oplus B$.

Furthermore, the bounded finite-one degree of a Martin-Löf random set A contains an infinite antichain of one-one degrees where the representatives are $B_x = A \oplus \{y : c(x,y) \in A\}$.

▶ Remark 20. It might be important to note that nonrecursive nonirreducible finite-one degrees contain an antichain of bounded finite-one degrees. To obtain this result, one modifies Theorem 8 and its proof as follows: First, in the numbering of all φ_e, one equips the functions $\varphi_1, \varphi_2, \ldots$ with a boundedness-function *bound* such that for each y there are at most $bound(e)$ many x with $\varphi_e(x) = y$. Note that introducing this bound makes the numbering nonacceptable, but it still covers all reductions — if for the first instance of the function, $bound(e)$ was too small, then one produces another index e' where then $bound(e')$ is bigger. Functions intending to violate their bound are forced to be partial in order to preserve the bound. In the light of this practice, one could even just set $bound(e) = e$.

Furthermore, in Algorithm 9 of the proof, the corresponding sentence is adjusted to the following form:

> While the number of x with $h_i(x) = y$ is less or equal to the product of $bound(e)$ and the number of x' with $h_j(x') \leq y$ do begin select the least x where $h_i(x)$ is not yet defined and let $h_i(x) = y$ end.

With the adjustments following from this change in the proof, the proof then actually shows that every recursive partial order can be embedded effectively into every nonrecursive and nonirreducible finite-one degree with respect to the bounded finite-one degrees inside that degree and not just with respect to the one-one degrees inside it. The recursive finite-one degrees coincide each with one bounded finite-one degree, thus they form a special case, as two of these degrees have infinitely many one-one degrees inside them.

5 Relativised finite-one equivalence classes inside many-one degrees

The previous sections investigated the structure of finite-one degrees inside many-one degrees and further results relating to the structure of bounded finite-one degrees and one-one degrees inside a finite-one degrees. However, one main topic is left to future research: how many finite-one degrees can there be in a many-one degree and how the structure of these finite-one degrees inside the many-one degree looks in detail. Therefore, this section is devoted to shed additional light on this question by introducing a further tool of investigation: Studying the order of degrees inside many-one degrees with respect to stronger reducibilities relativised to an oracle, mainly the halting problem K. This is made more precise in the following definition.

▶ Definition 21. A is C-one-one reducible to B if there is a C-recursive one-one function f with $A(x) = B(f(x))$ for all x. Similarly for C-finite-one reducibility. The number of C-finite-one equivalence classes of a many-one degree is called the C-finite-one size of the many-one degree.

Note that if C is recursive and the C-finite-one size of a many-one degree is one, then this many-one degree consists of a single one-one degree. This definition can also be used to show that no oracle C is strong enough to collapse all many-one degrees into finitely many equivalence classes; instead there always many-one degrees which have, even relative to C, C-finite-one size $\aleph_0$ and an antichain can be embedded into the equivalence classes relative to C inside the unrelatived many-one degree.

▶ **Theorem 22.** *Let C be any oracle and A be a set which is Martin-Löf random relative to C. Then there is an infinite antichain of finite-one degrees inside the many-one degree of A and these finite-one degrees are even pairwise C-finite-one incomparable.*

Proof. It is known from algorithmic randomness that if A is Martin-Löf random relative to C then there cannot be an C-r.e. equivalence relation $\sim$ such that for infinitely many numbers x there is an $y \neq x$ with $y \sim x$ and for all x, y it holds that $x \sim y$ implies $A(x) = A(y)$. The intuitive reason is that such an equivalence relation would allow to define a randomness test which captures A and thus proves that A is not random relative to C; more precisely the class S_k of all sets which respect the first k enumerated equivalences $\sim$ between two elements not yet made equivalent by the transitive closure of the previously enumerated equivalences has the measure 2^{-k} and all these classes S_k contain the set A. So by Martin-Löf's definition, A could not be random relative to C. Now when one constructs the set B_i via a h_i which maps infinitely many x to any y which is a multiple of 2^i but not of 2^{i+1} and only one x to each other y produces an array where, when i, j are different, any function f which C-recursively finite-one reduces B_i to B_j must implicitly map all each y with infinitely copies in B_i to either at least one y' with infinitely many copies in B_j or to infinitely many y'' with one copy in B_j only; thus the equivalence relation $\sim$ defined by $y \sim y'$ iff an x is mapped to x' with $h_i(x) = y$ and $h_j(x') = y'$ will create infinitely many linkages between pairs (x, x') if the mapping from B_i to B_j is a C-recursive finite-one reduction, thus such a mapping does not exist. Therefore the B_i as defined above represent an infinite antichain of finite-one degrees inside the many-one degree of A. ◀

For the next results, one fixes the oracle C to be the halting problem K. Note that all nonrecursive but K-recursive many-one degrees then have K-finite-one size 1. Furthermore, the greatest recursive many-one degree consists of three finite-one degrees, both unrelativised and also relative to K, that is, the K-finite-one size is also 3. So the next results aim to construct many-one degrees with K-finite-one sizes 2 and 4. Furthermore, one defines that for subsets X, Y of $\mathbb{N}$, $X \times Y$ equals the set $\{c(v, w) : v \in X \wedge w \in Y\}$ where $c(v, w)$ is Cantor's pairing function $0.5 \cdot (v + w) \cdot (v + w + 1) + w$. Cantor used this pairing function to show that finite products of the natural numbers have the same cardinality as the set of natural numbers while the set of real numbers has a higher cardinality; first results about this are laid out by him in the year 1874 [4].

▶ **Theorem 23.** *There is a many-one degree with K-finite-one size* 2.

Proof. Note that there is, relative to K, a K-maximal set. That set is K-r.e. and satisfies that every further K-r.e. set contains either only finitely many or all but finitely many elements of its complement. Now one can split this K-maximal set using a relativised version of Sacks' splitting theorem [25] into two K-r.e. sets whose Turing degrees relative to K are incomparable, call these two sets E_1 and E_2 and let E_3 be the complement of $E_1 \cup E_2$. Note that E_1 and E_2 are recursively inseparable relative to K. By a relativised version of the hyperimmune-free basis theorem of Jockusch and Soare [12], there is a set A such that, relative to K, A is of hyperimmune-free Turing degree and $E_1 \subseteq A$ and $E_2 \cap A = \emptyset$. Then E_3 has infinitely many elements inside A and infinitely many outside A, as otherwise A would be either a finite variant of E_1 or a finite variant of the complement of E_2 and both have K-r.e. Turing degree strictly above K and thus are not hyperimmune-free relative to K.

Assume that h is a many-one reduction from a set B to A and assume that B is many-one equivalent with A. Now the range of h can omit infintitely many elements of E_1 and of E_2, however, it must have infinite intersections with all of E_1, E_2, E_3, as otherwise the Turing degree of B is K-r.e. in contradiction to the choice of A. Now, by the fact that infinitely many elements of E_3 are in the range of h, almost all elements of E_3 must be in the range of A and furthermore, either almost all or only finitely many of them satisfy that infinitely many x are mapped to them. In the case that almost all elements of E_3 are only the image of finitely many x. Furthermore, let $g(x)$ be a K-recursive function such that, for infinitely many $y \in E_1 \cap range(h)$, $g(y)$ is bigger than the time to enumerate y into E_1 relative to K and for infinitely many $y \in E_2 \cap range(h)$, $g(y)$ is bigger than the time to enumerate y into E_2 relative to K and this property also holds for the finite variants E_4, E_5, E_6, E_7 of E_1 and E_2, respectively, defined below.

First one considers the latter case and one considers E_4 and E_5 finite variants of E_1 and E_2, respectively, which got the elements y of E_3 which have infinitely many preimages moved over to E_4 in the case that $y \in A$ and to E_5 in the case that $y \notin A$. So E_4 is a subset of A and E_5 is disjoint to A.

Now one can construct a new mapping h' relative to the oracle K using the many-one reduction h from B to A which will be modified as follows:

1. If $h(x)$ is enumerated into E_4 within $g(h(x)) + x$ steps then map x to the first element of E_4 not yet in the range of h' else let $h'(x) = h(x)$.
2. If $h(x)$ is enumerated into E_5 within $g(h(x)) + x$ steps then map x to the first element of E_5 not yet in the range of h' else let $h'(x) = h(x)$.
3. $h'(x) = h(x)$ for the remaining x.

Assume that h' maps infinitely many x to the same y. This can only happen if $h(x) = y$ for almost all x with $h'(x) = y$, as the first two cases assign each $y \in E_4 \cup E_5$ only once as a new element of h' which differs from that assigned by $h(x)$. However, if $h(x) = y$ for the same y and infinitely many x, then $y \in E_4 \cup E_5$ and is eventually enumerated into E_4 or E_5 and from then onwards always redirected to a new element outside the so far constructed range of h'. Therefore the new function is a K-finite-one reduction from B to A.

The other case is that for almost all $y \in E_3$, infinitely many x are mapped to y by h. Now one adjusts E_1 and E_2 to E_6 and E_7 by moving the finitely many elements $y \in E_3$ which are only the images of finitely many y over to E_6 and E_7; note that E_6 is a subset of A and E_7 is disjoint to A. Furthermore, consider the set $A \times \mathbb{N}$. Now one can construct a new mapping h'' relative to the oracle K with the following properties:

1. If x is enumerated into E_6 within $g(h(x)) + x$ steps relative to K then $h''(x)$ is the first element of $E_6 \times \mathbb{N}$ not yet in the range of h'';
2. If x is enumerated into E_7 within $g(h(x)) + x$ steps relative to K then $h''(x)$ is the first element of $E_7 \times \mathbb{N}$ not yet in the range of h'';
3. The remaining x are mapped in a bijective way to $c(h(x), z)$ for the first z where $c(h(x), z)$ is not yet in the range of h''.

Thus B is in this case now finite-one equivalent relative to K to the cylinder $A \times \mathbb{N}$; the surjectivity is coded into the mapping and furthermore each image $c(y, z)$ of a pair can only occur in the range of h'' at most once. Furthermore, A is strictly below $A \times \mathbb{N}$, as a K-recursive function can map only for finitely many values $c(y, z)$ with $y \in E_3$ to an $y' \in E_3$ with $y' > y$; otherwise E_3 would not be cohesive relative to K. Thus for each $y \in E_3$ exists a z with $c(y, z)$ being mapped to $E_1 \cup E_2$ and that would mean, that a finite-one reduction from $A \times \mathbb{N}$ to A would provide an algorithm to decide A relative to K, in contradiction to the fact that E_1 and E_2 have incomparable Turing degrees strictly above K while A has a Turing degree strictly above K which is also hyperimmune-free relative to K. ◀

▶ **Corollary 24.** *(a) Let E be a maximal set relative to K. Then the many-one degree of E has K-finite-one size* 3.

(b) Let A as in Theorem 23 and B be its complement. Then the many-one degree of $A \oplus B$ has the K-finite-one size 4.

Proof. The main ideas of the proof are the below ones; the details are similar to Theorem 23 and left to the reader.

(a) Let E be a set which is maximal relative to the oracle K. Then the three sets $E \times \mathbb{N}$, $E \oplus \emptyset$ and E itself satisfy that every set in the many-one degree of E is K-finite-one equivalent to one of these sets. For the verification, note for each set F many-one reducible to E via f and inside the many-one degree of E, the set G of all y with only finitely many x satisfying $f(x) = y$ is recursively enumerable relative to K and thus G contains either only finitely many or almost all members of the complement of E. Thus E is K-finite-one equivalent those F where G contains the complement of E and $E \oplus \emptyset$ is K-finite-one equivalent to those F where G contains almost all but not all members of the complement of G and $E \times \mathbb{N}$ is K-finite-one equivalent to those F where G contains only finitely many members of the complement of E, this case includes the case that G contains no members of the complement of E at all.

(b) Taking A as in the proof of Theorem 23 and letting B be the complement of A the proof is based on the adjustment of the proof of Theorem 23 to show that the many-one degree $A \oplus B$ satisfies that all members are K-finite-one equivalent to exactly one of the sets $A \oplus B$, $(A \times \mathbb{N}) \oplus B$, $A \oplus (B \times \mathbb{N})$ and $(A \times \mathbb{N}) \oplus (B \times \mathbb{N})$. ◀

Note that the A from Theorem 23 is not K-recursive and that it falls into two K-finite-one equivalence classes. The following result shows that one can obtain that the K-one-one equivalence classes of this many-one degree are linearly ordered and infinitely many.

▶ **Theorem 25.** *The set A from Theorem 23 satisfies that its many-one degree consists of infinitely many K-one-one equivalence classes and that for all K-one-one nonequivalent sets B, C in the many-one degree of A, either B is K-one-one reducible to C or C is K-one-one reducible to B.*

Proof. Let E_1, E_2, E_3 as in Theorem 23, but reuse the further sets of the E-series with a new meaning. Furthermore, let h_B and h_C be many-one reductions from B and C to A, respectively.

Let $B = A \times \{0, 1, \ldots, k, k+1\}$ and $C = A \times \{0, 1, \ldots, k\}$. The sets B and C are all many-one equivalent to A. Furthermore, one easily sees that C is one-one reducible to B, $c(x, y)$ with $y \leq k$ is mapped to itself and $c(x, y)$ with $y > k$ is mapped to $c(x, y+1)$ where c is Cantor's pairing function used in the definition of $\times$. It is easy to see that this reduction is correct. Now assume by way of contradiction that f is a K-recursive one-one reduction from B to C. Then for each x there is an $y \in \{0, 1, \ldots, k, k+1\}$ such that $f(c(x, y)) \notin \{x\} \times \{0, 1, \ldots, k\}$ by the pigeonhole principle. If $f(c(x, y)) = c(x', z)$ for some $z > k$ then one knows that $A(x) = 0$. If $f(c(x, y)) = c(x', z)$ for some $z \leq k$ then one knows that $A(x') = A(x)$ and $x' \neq x$. Furthermore, as the K-r.e. infinite set E_1 has an infinite K-recursive subset, one can define a K-recursive one-one function from $\mathbb{N}$ to $\mathbb{N}$ such that $f(x) \neq x$ and $A(x) = A(f(x))$ for all x. However, as $E_1 \cup E_2$ is K-r.e., either finitely many $x \in E_3$ satisfy $f(x) \in E_1 \cup E_2$ or all but finitely many $x \in E_3$ satisfy $f(x) \in E_1 \cup E_2$. Say the first. Then almost all $x \in E_3$ satisfy $f(x) \in E_3$. Now either almost all $x \in E_3$ satisfy $f(x) > x$ or almost all $x \in E_3$ satisfy $f(x) < x$. In the first case E_3 has an K-recursive infinite subset what is not the case. In the second case, E_3 has an infinite retraceable subset. For this one can define levels how often one can replace x by $f(x)$ until either $f(x) \notin E_3$ or $f(x) > x$, both happens only finitely often and thus these values can be recognised by a finite table. Thus one can count with oracle K how many applications of f lead to these finite values and obtain two K-r.e. infinite sets F_1 and F_2 which both intersect E_3 in contradiction to its choice. Thus f must map almost all x from E_3 to outside E_3. Now one can follow f as long until one either enumerates x or one of the images $f(x), f(f(x)), \ldots$ into E_1 or into E_2. This would allow to give a K-recursive decision procedure for A which is defined on all but at most finitely many numbers (those are members of E_3 on finite cycles inside E_3) and that contradicts to E_1 and E_2 being recursively inseparable relative to K.

Thus there is an infinite ascending chain of one-one degrees inside the many-one degree of A such that even the oracle K is not strong enough to allow to reverse the one-one reductions in this chain. Thus there are infinitely many K-one-one equivalence classes inside the many-one degree of A.

The next step is to show that the one-one degrees inside the many-one degrees are linearly preordered by K-one-one reductions, note that as just seen there are still infinitely many one-one degrees which are not made to coincide relative to K. Now let B, C be K-one-one inequivalent sets in the many-one degree of A. Let $fin_B = \{y$: only finitely many x satisfy $h_B(x) = y\}$ and $fin_C = \{y$: only finitely many x satisfy $h_C(x) = y\}$.

Now assume that $fin_C \cap E_3$ is finite. Then B is K-one-one reducible to C as follows. First one sees that C is K-finite-one equivalent to $A \times \mathbb{N}$ by Theorem 23. The K-finite-one reduction from $A \times \mathbb{N}$ to C can be made one-one by a better book-keeping of this reduction. To see this, note that if f is the K-recursive finite-one reduction from $A \times \mathbb{N}$ to C, then for every y the set $\{f(c(y,z)) : z \in \mathbb{N}\}$ is infinite and thus one can make f on this domain to be one-one, as for all z, z' it holds that $A(f(c(y,z))) = A(f(c(y,z')))$. This adjustment can be done uniformly in y and thus A is K-one-one reducible to C. Now B is one-one reducible to $A \times \mathbb{N}$, as that is a cylinder. Thus B is K-one-one reducible to C by concatenating the two K-one-one reductions.

The case that $fin_B \cap E_3$ is finite allows for symmetric reasons that C is K-one-one reducible to B.

The remaining case is that $fin_B \cap E_3$ and $fin_C \cap E_3$ are both infinite. Let $E_4 = E_3 \cap fin_B \cap fin_C$, E_4 is a finite variant of E_3 by E_3 being cohesive relative to K, those of the finitely many left-over elements in $E_3 - E_4$ which are in A are put into E_1 and those not in A are put into E_2. Now let $E_5 = \{y \in E_4$: there are at most as many x with $h_B(x) = y$ as z with $h_C(z) = y\}$ and $E_6 = E_4 - E_5$. One of E_5 or E_6 is finite and the elements of that set will be moved into either E_1 or E_2 depending on their membership in A as before. Thus assume without loss of generality that E_5 is the infinite set, so one will show that B is K-one-one reducible to C. Note that E_5 is cohesive relative to K like E_4.

Now let $D_1 \subseteq \{x : h_C(x) \in E_1\}$ and $D_2 \subseteq \{x : h_C(x) \in E_2\}$ be infinite K-recursive sets. To see this, assume that one of them, D_2, does not exist. Then the set $\{x : h_C(x) \in E_2\}$ must be finite. Furthermore, as E_5 is cohesive relative to K, the range of h_C contains either finitely many or all but finitely many elements of E_5. If now the intersection of the range of h and E_5 is finite, then almost all elements of C are mapped into A and therefore C has a recursive complement and is recursive itself, a contradiction, as C belongs to a nonrecursive many-one degree. If the union of E_1 and the range of h is a finite variant of $E_1 \cup E_5$ then E_2 is a finite variant of the complement of a K-r.e. set in contrast to be itself K-r.e. and strictly above K. Thus D_1 and D_2 both exist and can be used to show that B is K-one-one reducible to C. Now for every y at least one of the following three conditions holds:

1. $y \in E_1$;
2. $y \in E_2$;
3. There are only finitely many x with $h_B(x) = y$ and finitely many z with $h_C(z) = y$ and there are at least so many such z as such x.

One can investigate for each y using oracle K the situation until one of these three conditions is found to apply, these are three K-r.e. conditions which cover the set of all y and thus one will eventually be found to hold.

Now for all x one computes $f(x)$ for a K-one-one reduction from B to C by doing the following: One computes $y = h_B(x)$ and uses the oracle K to find out which of the three conditions above holds; if y had been already used before dealing with x, one recovers the decision which was made for y from a log and then proceed to the same case as previously.

1. If $y \in E_1$ applies, then one picks the first element $z \in D_1$ found not yet in the range of f and defines $f(x) = z$.
2. If $y \in E_2$ applies, then one picks the first element $z \in D_2$ found not yet in the range of f and defines $f(x) = z$.
3. If the third condition applies then one searches for a z not yet in the range of f with $h_C(z) = y$ and let $f(x) = z$ for this z.

The so constructed f is one-one, K-recursive and reduces B to C. ◀

6 Conclusion

The present work studies the collection of one-one degrees inside finite-one degrees and the collection of finite-one degrees inside many-one degrees. For finite-one degrees, it is shown that they consist of a single one-one degree if they are the greatest finite-one degree in their many-one degree; otherwise they consist of infinitely many bounded finite-one degrees and contain an antichain of these which implies that they also contain an antichain of one-one degrees. This solves an open problem which was around since Young [31] embedded a dense linearly ordered set into all nonrecursive many-one degrees which consist of several one-one degrees; Odifreddi [20, 21] stated this problem explicitly as open. Progress towards this problem was done by Dëgtev [7] and Batyrshin [1], who proved the existence of antichains in all r.e. nonrecursive nonirreducible many-one degrees and all limit-recursive nonrecursive nonirreducible many-one degrees, respectively.

The present results answer this question in full generality and affirmatively. Furthermore, the paper initiates a detailed study of the finite-one degrees and bounded finite-one degrees which are between one-one-degrees and many-one degrees so that the following inclusion relation between these degrees hold (for the degrees of a fixed set):

$$\text{One-one degree} \subseteq \text{bounded finite-one degree} \subseteq \text{finite-one degree} \subseteq \text{many-one degree.}$$

The following paragraphs (a), (b) and (c) give an overview of the results of the present work.

(a) Finite-one degrees inside many-one degrees. Every many-one degree consists of at least one and up to countably many finite-one degrees; among those is a greatest finite-one degree which coincides with its one-one degree (that is, it is an irreducible one-one degree) and the many-one degree is irreducible if and only if it has only one finite-one degree. The recursive many-one degree consists of three finite-one degrees which are those of all finite nonempty sets, all cofinite sets with a nonempty complement and all other recursive sets; the third degree is irreducible and the other two degrees coincide with bounded finite-one degrees which in turn are ascending chains of one-one degrees order-isomorphic to the natural numbers and $<$. Theorem 8 shows that nonrecursive and nonirreducible many-one degrees have only one irreducible finte-one degree inside them, namely the greatest one, and furthermore at least one nonirreducible finite-one degrees, they are described under (b). Their number is at least one and at most countable, see the open problems below for more information. Using relativisation, the question into how many relativised finite-one equivalence classes a many-one degree can fall sheds some initial light into this question and

Corollary 24 shows that the number of these can be one, two, three, four and infinite relative to K; for five and greater, no construction is given but the authors believe that at least some of these numbers of equivalence classes are also possible; in order to limit the size of this paper, the study was not extended. Furthermore, Remark 15 points out that the irreducible many-one degrees can be characterised as those consisting of exactly one finite-one degree, thus providing one possible characterisation for those requested by Odifreddi [20, Problem 4] who asked for criteria characterising when many-one degrees are irreducible/nonirreducible.

(b) One-one degrees and Bounded finite-one degrees inside finite-one degrees. Irreducible finite-one degrees coincide with their one-one degree. Nonrecuirsive nonirreducible finite-one degrees satisfy that they embed antichains and all other recursive partial orders by Theorems 2 and 8. These results in particular show that for nonrecursive and nonirreducible finite-one degrees, they allow to embed by a uniformly recursive family of invertible finite-one reductions a sequence of one-one degrees which are ordered according to any given recursive partial order; one can furthermore achieve that the so embedded one-one degrees are all not bounded finite-one equivalent with each other, see Remark 20. As every nonirreducible and nonrecursive many-one degree contains an nonirreducible finite-one degree, this answers the question of Odifreddi [20, Problem 4].

(c) One-one degrees inside bounded finite-one degrees. The partial orders of one-one degrees inside finite-one degrees (by one-one reducibility) can allow more variety than in the case of finite-one or many-one degrees in the sense that more different cases arise. Theorem 18 shows that the one-one degrees inside a bounded finite-one degree can be infinitely many which are linearly ordered — the order type is that of the natural numbers with $<$. Furthermore, there are bounded finite-one degrees which have finite but no infinite antichains and those which have infinite antichains.

Open Questions. Though much progress to the understanding of the structure inside many-one degrees is made, still many questions are open.

1. Does every nonrecursive many-one degree have a least finite-one degree inside?
2. A finite-one degree is a maximal finite-one degree inside a given many-one degree if it is strictly below the greatest finite-one degree, but there are no further finite-one degrees between these two. A finite-one degree is a minimal finite-one degree inside a given many-one degree if it is not the least finite-one degree inside its many-one degree and furthermore either there is no or only the least finite-one degree below this degree. Let the triple (h, i, j) denote that a many-one degree has h least, i minimal and j maximal finite-one degrees. Which triples occur, that is, belong to some many-one degree? Irreducible many-one degrees contribute $(1, 0, 0)$ and the greatest recursive many-one degree contributes $(0, 2, 2)$ to the possible combinations (h, i, j).
3. For which natural numbers ℓ do there exist many-one degrees consisting of exactly ℓ finite-one degrees? So far known numbers are 1 (irreducible many-one degrees) and 3 (witnessed by the greatest recursive many-one degree).
4. For which natural numbers ℓ do there exist many-one degrees consisting of ℓ K-finite-one equivalence classes? The irreducible many-one degrees as well as the limit-recursive nonrecursive many-one degrees provide $\ell = 1$. Furthermore, Theorem 23 provides $\ell = 2$ and Corollary 24 provides $\ell = 3$ and $\ell = 4$.
5. For the K-finite-one equivalence classes, one can also ask which combinations (h, i, j) for least, minimal and maximal K-finite-one equivalence classes inside a given many-one degree can occur. Besides $(1, 0, 0)$ and $(0, 2, 2)$ from above, one gets also $(1, 1, 1)$ from the many-one degree with 2 K-finite-one equivalence classes and the many-one

degree with four K-finite-one equivalence classes in Corollary 24(b) is actually a diamond contributing $(1, 2, 2)$ to the list of possible triples.

6. Theorem 18 provides a bounded finite-one degree entirely consisting of linearly ordered one-one degrees forming an ascending chain. Can one also have other linear orders than this specific one of the natural numbers?

The construction of Theorem 18 was attempted to be done more general—Stephan [28] studied important questions about strong degrees and wanted in this paper also address other open questions about strong degrees including the one addressed in the current paper. His intention was to construct a whole many-one degree where the the one-one degrees are linearly ordered and together with Zhang—during his exchange to Singapore as an undergraduate student—he tried another time to make this construction work. Now Theorem 2 shows that generalising Theorem 18 to finite-one or many-one degrees is impossible. So the open question is which linear orders can be realised by the set of all one-one degrees inside a bounded finite-one degrees and whether there are any besides the one-element linear order and the linear order of natural numbers. The remarks after the theorem provide other partial orders which are the orders of a bounded finite-one degree, but no further linear order.

Note that finite-one degrees and bounded finite-one degrees are less investigated than one-one degrees and many-one degrees, thus more questions are open for these than for the other two types of degrees; Bjørn Kjos-Hanssen, one of the authors of the article [13] as well as the authors of this paper are not aware of any recursion-theoretic paper studying finite-one degrees besides the mentioned paper, though the notion might be implicitly used in some proofs. However, finite-one functions are used frequently in topology and set theory to define reductions in their fields.

There are two parallels between the study of numberings and very strong degrees: First the same type of reducibilies as considered in the present paper (one-one, bounded finite-one, finite-one, many-one) can be used in the field of study of numberings; however, there are differences between these fields: While sets have only two cases which can occur ("$x \notin A$", "$x \in A$"), for numberings there are usually infinitely many objects, say functions between natural numbers or subsets of the natural numbers, with additional side conditions restricting these objects to countably many. Therefore one can construct and study Friedberg numberings [8, 9] and other special numberings which cannot be defined in the case of comparing single subsets of natural numbers. However, one has certain special sets whose many-one degrees and related subdegrees have special properties worth further investigation. Note that among the four reducibilities between numberings, the many-one reducibility is the standard reducibility investigated and the other three are less often considered (if at all).

The second parallel is that the proof methods of Theorem 2 and 8 construct arrays of one-one degrees inside the many-one degree of A by constructing families of invertible finite-one reductions to A — these reductions are constructed without any knowledge about A beyond these: A is neither finite nor cofinite nor a cylinder. Only the verification which makes the functions f_d with cofinite domain total in the case that the underlying φ_e is a one-one reduction between B_i and B_j which should not exist, uses finite amount of knowledge about A in order to make the f_d a total strictly ascending self-reduction. Such a self-reduction only exists when A is a cylinder, as that was a priori excluded, there are no one-one reductions φ_e between distinct B_i, B_j in Theorem 2. This type of handling objects enumerated without really knowing what they are has also been the practice in various papers within the theory of numberings, for example, the work of Goncharov, Lempp and Solomon [9].

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