Computability, Consistency, and a Theorem on Fractals

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Classical mathematics can be studied via mathematical logic.

In this talk: set theory and computability theory affect geometric measure theory.

Goal of this talk

An example a classical theorem in geometric measure theory which is sensitive to set-theoretical axioms and computability-theoretical concepts.

Turing Computability

Work over $\omega = \{0, 1, 2, \ldots\}$.

Definition

A set A is (Turing) computable if there exists a program P which halts in finite time and outputs

$$P(n) = \begin{cases} \text{yes} & \text{if } n \in A \\ \text{no} & \text{if } n \notin A. \end{cases}$$

We can let a program have access to oracles:

Definition

A program P is an oracle program for A if it can ask at any point whether "$n \in A$". Write P^A.

Definition

A set A (Turing) computes B if there exists a program P^A which computes B.

Write $B \leq_T A$. This is a partial ordering. The equivalence classes are called Turing degrees. $\mathbf{0}$ is the degree of computable sets. By diagonalising against all programs one gets the jump $\mathbf{0}'$, and so

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \dots$$

Every set A has a degree $\mathbf{a}$, and jumps $\mathbf{a}'$, $\mathbf{a}''$,...

- $|\{B \mid B \leq_T A\}| = \aleph_0$ for every A.
- There are $2^{\aleph_0}$ Turing degrees.
- Every countable partial order is embeddable into degrees.
- $\mathbf{0}^{(\alpha)}$ exists for (some) ordinals $\alpha < \omega_1$. $\rightarrow$ hyperarithmetic sets

In classical mathematics: there is no program which, on input of any diophantine equation with $\mathbb{Z}$-coefficients, can compute whether a $\mathbb{Z}$-solution exists. (MRDP)

A Cauchy name for $x \in \mathbb{R}$ is a sequence $(x_i)_i$ of rationals which converges quickly: $(\forall n)(\forall m \geq n)(|x_m - x| < 2^{-n})$.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there exists P such that: whenever $(x_i)_i$ is a Cauchy name of $x \in \mathbb{R}$ then

$$P^{(x_i)_i}(n) = q_n \in \mathbb{Q} \quad \text{such that} \quad |q_n - f(x)| < 2^{-n}.$$

Theorem

Every computable function is continuous.

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Theorem

Every computable function is continuous.
Every continuous function is computable relative to some oracle.

Sets of reals

Not only functions between reals can be analysed, but also sets of reals. Topologically, we get the Borel hierarchy:

$\boldsymbol{\Sigma}^0_1 =$ open sets $\qquad\qquad \boldsymbol{\Pi}^0_1 =$ closed sets

$\boldsymbol{\Sigma}^0_\alpha =$ union of $\boldsymbol{\Pi}^0_\beta$-sets $\qquad\qquad \boldsymbol{\Pi}^0_\alpha =$ intersection of $\boldsymbol{\Sigma}^0_\beta$-sets

$\boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$

where $\beta < \alpha < \omega_1$.

$$
\begin{tikzcd}
 & \boldsymbol{\Sigma}^0_1 \arrow[dr, phantom, sloped, "\subset"] & & \boldsymbol{\Sigma}^0_2 & & & \boldsymbol{\Sigma}^0_{\alpha+1} \arrow[dr, phantom, sloped, "\subset"] & & \\
\boldsymbol{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & \boldsymbol{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & \cdots & \boldsymbol{\Delta}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & \boldsymbol{\Delta}^0_{\alpha+2} & \cdots \\
 & \boldsymbol{\Pi}^0_1 \arrow[ur, phantom, sloped, "\subset"] & & \boldsymbol{\Pi}^0_2 & & & \boldsymbol{\Pi}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subset"] & &
\end{tikzcd}
$$

Borel sets are constructible, and this can be expressed in terms of Turing computability...

Theorem

A set $A \subseteq \mathbb{R}$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\alpha}$ iff there exists P and an oracle Z such that for all $x \in \mathbb{R}$ we have

$$x \in A \iff P^{(x \oplus Z)^{(\alpha)}} \text{ halts.}$$

Proof of case $\alpha = 0$.

Let A be open, and consider $\{(q_i, r_i) \mid i < \omega, q_i, r_i \in \mathbb{Q}\}$.
Let $f : \omega \to \omega$ be so that $\operatorname{ran}(f) = \{i < \omega \mid (q_i, r_i) \subset A\}$. Some Z computes f via some program P^Z. Run P^Z to compute $f(0), f(1), \ldots$. Whenever P^Z outputs i, check if $q_i < x < r_i$. This check takes finite time.
If such a Z and P exist, then for every x the program $P^{(x \oplus Z)}$ can tell in finite time whether $x \in A$. Hence it only reads finitely many bits of x in the process, and hence A must be open. $\square$

Consistency and Provability

The Borel hierarchy can measure provability, by looking just beyond it.

Theorem (Souslin)

There exists a set that is not Borel.

Continuous images of Borel sets are called $\utilde{\mathbf{\Sigma}}^1_1$. This gives the projective hierarchy.

$$
\begin{tikzcd}[column sep=small]
 & \utilde{\mathbf{\Sigma}}^1_1 \arrow[dr, phantom, "\subset" sloped] & & \utilde{\mathbf{\Sigma}}^1_2 & \\
\text{Borel} = \utilde{\mathbf{\Delta}}^1_1 \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \utilde{\mathbf{\Delta}}^1_2 \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \cdots \\
 & \utilde{\mathbf{\Pi}}^1_1 \arrow[ur, phantom, "\subset" sloped] & & \utilde{\mathbf{\Pi}}^1_2 &
\end{tikzcd}
$$

Definition

A set $A \subset \mathbb{R}$ has the perfect set property if it is either countable or if it contains a perfect subset (i.e. a copy of Cantor space 2^ω).

Which sets satisfy the PSP?

Axioms	Behaviour
ZF	PSP holds for all $\boldsymbol{\Sigma}^1_1$ sets (Souslin)
ZFC	PSP fails for some set (Bernstein)
ZF + AD	PSP holds for all sets (Mycielski, Swierczkowski)
ZFC + ($V=L$)	PSP fails for some $\boldsymbol{\Pi}^1_1$ set (Gödel)

Gödel's theorem gives an optimal, or definable counterexample (over the base theory ZF).

A Theorem on Fractals

Hausdorff measure is an extension of Lebesgue measure, which can measure *all* sets of reals. Its coverings are given a weight:

- if the weight is too high, Hausdorff measure is zero
- if the weight is too low, Hausdorff measure is infinite.

The Hausdorff dimension of a set $A \subset \mathbb{R}^2$ is the critical value at which Hausdorff measure is just right.

Theorem (Marstrand, 1954)

Let $A \subset \mathbb{R}^2$ be $\utilde{\boldsymbol{\Sigma}}^1_1$. Then for almost every angle θ we have

$$\dim_H(\mathrm{proj}_\theta(A)) = \min\{1, \dim_H(A)\}.$$

Question

Can more be proven over ZFC*?*

Theorem (Davies, 1979)

(CH) *There exists a set $E \subset \mathbb{R}^2$ for which*

$$\dim_H(E) = 1 \text{ yet for every } \theta \text{ we have } \dim_H(\mathrm{proj}_\theta(E)) = 0.$$

The projective hierarchy can calibrate consistency:

Question

Davies' set is $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$. Is it consistent (over ZFC*) that a set simpler than $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ fails Marstrand's theorem?*

Theorem (R.)
($V{=}L$) *There exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subset \mathbb{R}^2$ *for which*

$$\dim_H(E) = 1 \text{ yet for every } \theta \text{ we have } \dim_H(\mathrm{proj}_\theta(E)) = 0.$$

Marstrand proved that every $\underset{\sim}{\mathbf{\Sigma}}^1_1$ set satisfies the theorem, hence a $\underset{\sim}{\mathbf{\Pi}}^1_1$ definable counterexample is optimal, and proves the exact consistency strength.

Our proof uses classical computability theory, descriptive set theory, and hyperarithmetic theory. As we prove consistency, we may assume $V{=}L$.

The proof

Kolmogorov complexity

A string $\sigma \in 2^{<\omega}$ has Kolmogorov complexity $K(\sigma) = n$ if

$$n = \min\{k < \omega \mid (\exists P)(\ell(P) = k \wedge P(\emptyset) = \sigma)\}.$$

- 0^{1000} has low complexity.
- Any initial segment of π (in binary) has low complexity.
- Random reals exist (by a counting argument).

For $x = 0.10110111\ldots$ define

$$x{\restriction}_5 = 0.10110$$

and $x{\restriction}_n$ similarly for any $x \in \mathbb{R}$, $n < \omega$.

Kolmogorov complexity controls Hausdorff dimension!

Theorem (Lutz and Lutz, 2018)

If $A \subset \mathbb{R}^2$ then

$$\dim_H(A) = \min_{Z \in 2^\omega} \sup_{x \in A} \liminf_{n \to \infty} \frac{K^Z(x\restriction_n)}{n}.$$

Form (the complexity of) points one can measure the complexity of sets—hence it's called the point-to-set principle.

Lemma

Every countable set has Hausdorff dimension 0.

Proof.

Suppose $A = \{x_i \mid i < \omega\}$. Let $Z = \bigoplus_i x_i$. Let P compute $x_i\restriction_n$ on input (i, n). For fixed i, the pair (i, n) can be coded in length $\log(n) + c$, which vanishes $/n$ as $n \to \infty$. $\square$

Sets of reals by recursion

Theorem (Erdős, Kunen and Mauldin; A. Miller; Vidnyánszky)

($V{=}L$) Recursive constructions of sets of reals can be carried out to build a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set if at every step of the recursion there exist arbitrarily complex[1] witnesses.

Call this the $\underset{\sim}{\mathbf{\Pi}}^1_1$ recursion theorem.

Can be used to (re-)prove the existence of $\underset{\sim}{\mathbf{\Pi}}^1_1$ Hamel bases, two-point sets, MAD families, etc.

We use this to build a set that fails Marstrand's theorem badly by using the point-to-set principle.

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[1] in the Turing degrees (or hyperdegrees)

The proof

Theorem (R.)

(V=L) There exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subset \mathbb{R}^2$ *for which* $\dim_H(E) = 1$ *yet for every* θ *we have* $\dim_H(\mathrm{proj}_\theta(E)) = 0$.

Proof

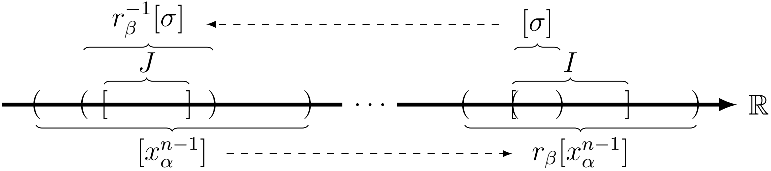
Iterate over all lines through the origin $\{N_\alpha \mid \alpha < \omega_1\}$. At stage α, suppose $E{\restriction_\alpha} = \{x_\beta \mid \beta < \alpha\}$ is given. Let $f : \omega \to \alpha$ be a surjection for which $f^{-1}[\{\beta\}]$ is infinite for every $\beta < \alpha$. Let r_β be the projection factor for N_β. Build x_α by baby Cohen forcing:

- x^0_α = empty string
- $x^n_\alpha \in D_{f(n)} = \left\{\sigma \,\middle|\, (\exists m)(K(\sigma r_{f(n)}{\restriction_m}) < 2^{-n})\right\}$ and $x^n_\alpha \succ x^{n-1}_\alpha$.

The meat of the proof lies in the task of showing that every D_β is dense in $2^{<\omega}$ *and* that there is space for coding.

Proof contd.

Recall $D_\beta = \{\sigma \mid (\exists m)(K(\sigma r_\beta\restriction_m) < 2^{-n})\}$, and let x_α^{n-1} be given. Work in terms of intervals: x_α^{n-1} induces an interval $[x_\alpha^{n-1}]$. Consider the interval $r_\beta[x_\alpha^{n-1}]$. Find an extension σ which ends in a long sequence of zeroes. Take the pull-back and ensure that the sequence of zeroes is preserved regardless of the extension.



Importantly, the length of σ can be computably bounded. Now, by the point-to-set principle and the $\underset{\sim}{\mathbf{\Pi}}^1_1$ recursion theorem, the set $E = \{x_\alpha \mid \alpha < \omega_1\}$ works. $\square$

By enumerating over suitable oracles (and more coding) one can show the following:

Theorem (R.)

(V=L) *For every* $\epsilon \in (0,1)$ *there exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subset \mathbb{R}^2$ *for which* $\dim_H(E) = 1 + \epsilon$ *yet for every* θ *we have* $\dim_H(\mathrm{proj}_\theta(E)) = \epsilon$.

This is optimal by classical facts of geometric measure theory.

Thank you