Regularity Properties in Fractal Geometry, Old and New

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Classical mathematics can be studied via mathematical logic.

In this talk: set theory and computability theory affect geometric measure theory.

Goal of this talk

- show how descriptive set theory can give explicit bounds on provability for regularity properties in fractal geometry *(Marstrand's theorem)*
- explain a useful connection between fractal geometry and computability theory

What does it mean to measure complexity?

Complexity = value in some stratified space

Examples

Classical mathematics:

- vector spaces $\to$ dimension
- finite groups $\to$ (length of) decomposition series
- subsets of $\mathbb{R}$ $\to$ dimension (Hausdorff, packing,...)

Mathematical logic:

- formulas $\to$ quantifier complexity
- subsets of ω $\to$ Turing degree
- subsets of $\mathbb{R}$ $\to$ Borel/projective hierarchy

Computability theory can bridge this gap!

Example: Regularity Properties

A regularity property is a property of sets of reals (i.e. elements of $\mathbb{R}$) which describe a “nice” structural behaviour.

Definition

A set $A \subseteq \mathbb{R}$ has the perfect set property if it is either countable or if it contains a perfect subset (i.e. a copy of Cantor space 2^ω).

So, no set with the PSP can be a counterexample to the Continuum Hypothesis.

Question

Which sets satisfy these regularity properties?
And how do we measure that?

Turing Computability

Work over $\omega = \{0, 1, 2, \ldots\}$. Main idea: successful computations take finite time and finite resources *(use principle)*.

Definition

A set $A \subseteq \omega$ is (Turing) computable if there exists a program P which halts in finite time and outputs

$$P(n) = \begin{cases} \text{yes} & \text{if } n \in A \\ \text{no} & \text{if } n \notin A. \end{cases}$$

Turing's insight: formalise access to more information via oracles:

Definition

A program P is an oracle program for $A \subseteq \omega$ if it can ask at any point whether "$n \in A$". Write P^A.

Definition

A set *A* (Turing) computes *B* if there exists a program P^A which computes B. Write $B \leq_T A$.

This is a partial ordering. Equivalence classes are the Turing degrees. $\mathbf{0}$ is the degree of computable sets. By diagonalising against all programs one gets the jump $\mathbf{0}'$:

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \dots$$

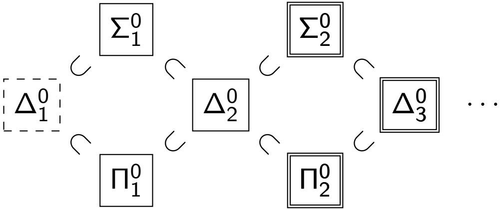
Σ^0_1-complete

Σ^0_2-complete

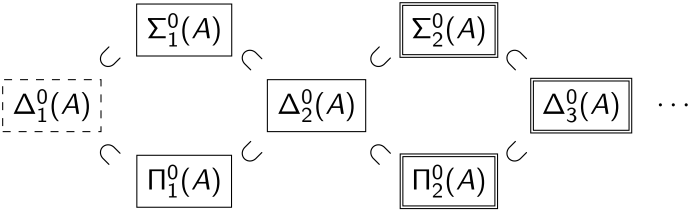
Completeness

Ask any Σ^0_1-question, i.e. search for a witness—it'll tell you the answer in finite time. Classical example for $\mathbf{0}'$: the halting problem.

This gives the arithmetical hierarchy over ω:



Here, $\Delta^n_0 = \Sigma^n_0 \cap \Pi^n_0$. For example, Δ^0_1 = c.e. + co-c.e.
This can be relativised to any set $A \subseteq \omega$:



Sets of reals

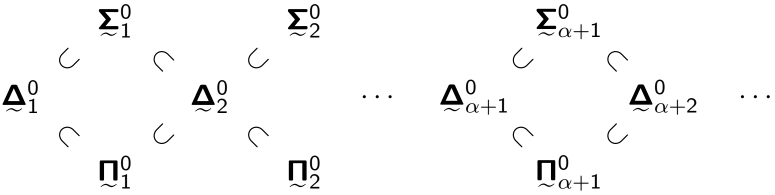
Not only sets of numbers can be analysed, but also sets of reals.
Topologically, we get the Borel hierarchy:

$\utilde{\boldsymbol{\Sigma}}^0_1$ = open sets $\qquad$ $\utilde{\boldsymbol{\Pi}}^0_1$ = closed sets

$\utilde{\boldsymbol{\Sigma}}^0_\alpha$ = union of $\utilde{\boldsymbol{\Pi}}^0_\beta$-sets $\qquad$ $\utilde{\boldsymbol{\Pi}}^0_\alpha$ = intersection of $\utilde{\boldsymbol{\Sigma}}^0_\beta$-sets

$\utilde{\boldsymbol{\Delta}}^0_\alpha = \utilde{\boldsymbol{\Sigma}}^0_\alpha \cap \utilde{\boldsymbol{\Pi}}^0_\alpha$

where $\beta < \alpha < \omega_1$.



Superscript 0 indicates first-orderness—this can be made explicit via Turing computability! Think: Borel = computable

Consistency and Provability

The Borel hierarchy can be extended to the right: there exists a set that is not Borel (Souslin). Continuous images of Borel sets are called $\boldsymbol{\Sigma}^1_1$—this gives the projective hierarchy.

$$
\begin{array}{ccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & & \\
\text{Borel} = \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & & \\
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & &
\end{array}
$$

(Think of $\boldsymbol{\Sigma}^1_1$ as c.e. with real witnesses.)

Note: The projective hierarchy is well-ordered!

Regularity Properties

Definition

A set $A \subseteq \mathbb{R}$ has the perfect set property if it is either countable or if it contains a perfect subset (i.e. a copy of Cantor space 2^ω).

Question

Which sets *provably* satisfy the PSP?

$$\begin{array}{ccccccccc} & & & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & \\ & & & \subset & & \supset & & \subset & & & \\ \text{Borel} = \boldsymbol{\Delta}^1_1 & & & & & & \boldsymbol{\Delta}^1_2 & & & & \cdots \\ & & & \supset & & \subset & & \supset & & & \\ & & & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & \end{array}$$

Some Axioms of Set Theory

ZF = Zermelo-Fränkel set theory

Some axioms give more sets:

AC = Axiom of Choice

- "every non-empty set has a choice function"
- equivalent with a host of axioms: every set can be well-ordered, Zorn's lemma, every vector space has a basis
- at the cost of definable structure: Vitali set, Banach-Tarski

Some axioms give more structure:

AD = Axiom of Determinacy

- "every two-player game on $\mathbb{R}$ has a winning strategy"
- provable for Borel sets (D. Martin), but not beyond; so every regularity property expressible as a game holds for Borel sets
- incompatible with the Axiom of Choice

Best of both worlds:

$(V{=}L)$ = Axiom of Constructibility

- "every set is constructible" (think "definable")
- proves the Axiom of Choice (and the generalised continuum hypothesis)

In $(V{=}L)$, we get *both* lots of sets (through AC) *and* a lot of structure (through definability of every set)!

This gives us the ideal environment to find optimal definable counterexamples to regularity properties.

The "usual" pattern for regularity properties

Axioms	Behaviour
ZFC	
ZFC	
ZF + DC + AD	
ZFC + $(V{=}L)$	

$$
\begin{array}{ccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 \\
 & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} & \\
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Axioms	Behaviour
ZFC	PSP holds for all $\boldsymbol{\Sigma}^1_1$ sets (Souslin)
ZFC	
ZF + DC + AD	
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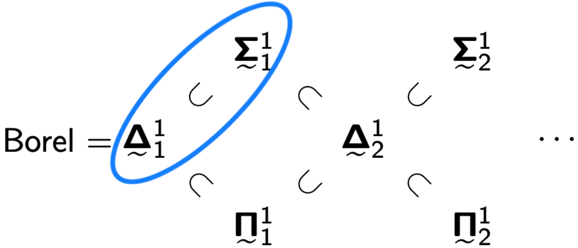

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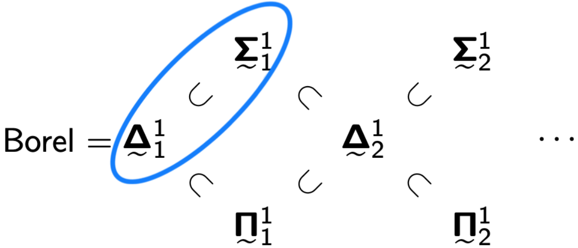

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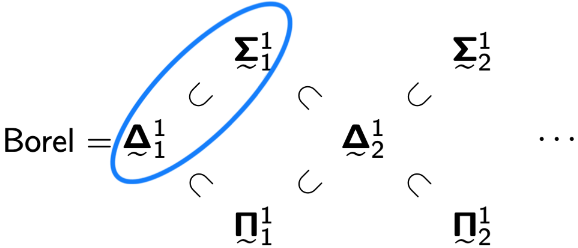
ZFC + (V=L)	



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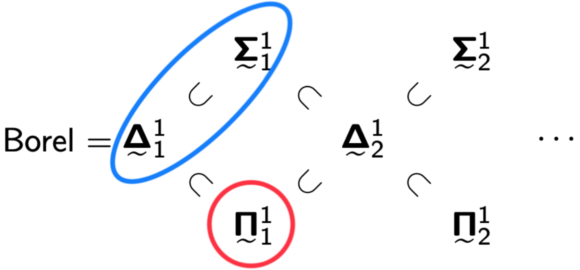
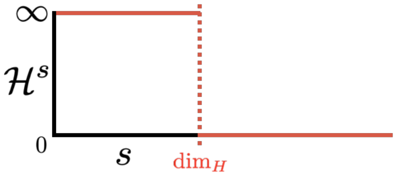
ZFC + $(V{=}L)$	PSP fails for some $\utilde{\boldsymbol{\Pi}}^1_1$ set (Gödel)



A Projection Theorem for Fractals

The s-dimensional Hausdorff outer measure $\mathcal{H}^s$ is a generalisation of Lebesgue outer measure; its coverings are given a weight:

- if s is too large, $\mathcal{H}^s$ is zero.
- if s is too small, $\mathcal{H}^s$ is infinite.



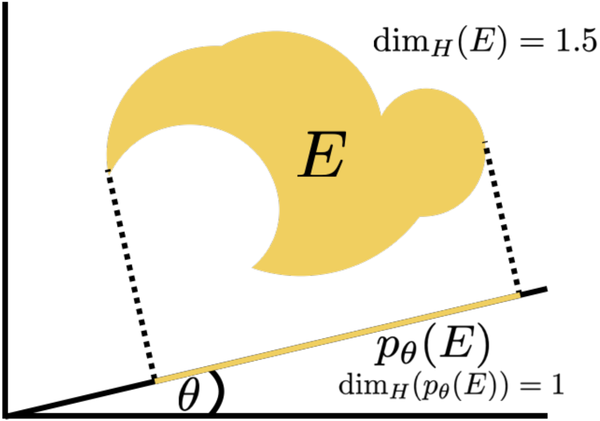
Example

- $\dim_H \left([0,1]^2\right) = 2$
- $\dim_H(\text{middle-third Cantor set}) = \log(2)/\log(3)$

Every set of reals has a Hausdorff dimension. $\dim_H$ a classical object of study in geometric measure theory.

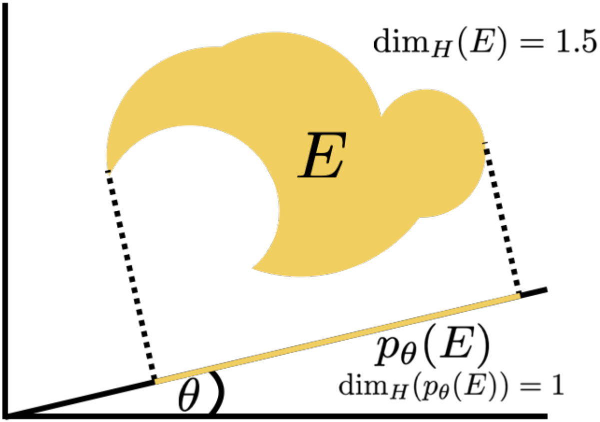
Definition

A set $A \subseteq \mathbb{R}^2$ has the Marstrand property if for almost every angle θ we have $\dim_H(\mathrm{proj}_\theta(A)) = \min\{1, \dim_H(A)\}$.



Theorem (Marstrand, 1954)

Every $\underset{\sim}{\mathbf{\Sigma}}^1_1$ *set has the Marstrand property.*



Can we prove *more* in ZFC?

Axioms	Behaviour
ZFC	
ZFC + CH	
ZF + DC + AD	
ZFC + $(V = L)$	

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\begin{array}{ccccccccccc}
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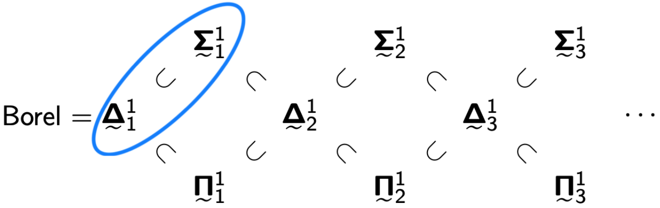
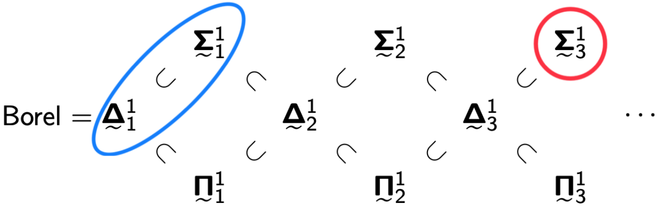
 & & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & & & \boldsymbol{\Pi}^1_3 & &
\end{array}
$$

Axioms	Behaviour
ZFC	MP holds for all $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets (Marstrand, 1954)
ZFC + CH	
ZF + DC + AD	
ZFC + $(V = L)$	



Axioms	Behaviour
ZFC	MP holds for all $\boldsymbol{\Sigma}^1_1$ sets (Marstrand, 1954)
ZFC + CH	MP fails for some set (Davies, 1979)
ZF + DC + AD	
ZFC + $(V{=}L)$	



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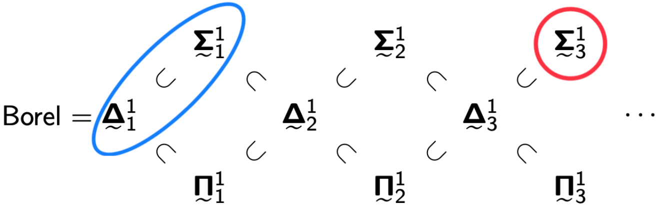

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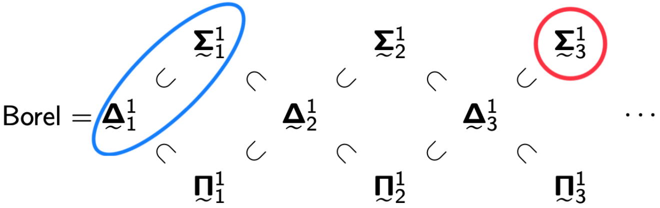
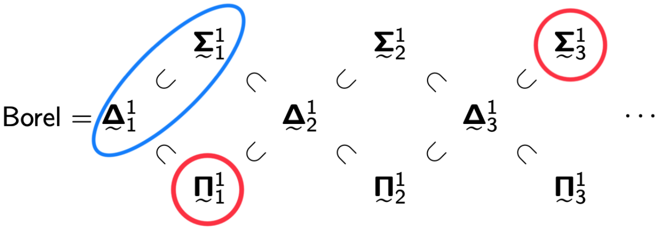


Completing the Picture for MP

Theorem (R.)

$(V{=}L)$ *There exists a* $\underset{\sim}{\boldsymbol{\Pi}}{}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which*

$$\dim_H(E) = 1 \text{ yet for every } \theta \text{ we have } \dim_H(\mathrm{proj}_\theta(E)) = 0.$$



How do we construct such a set? By recursion!

From Points to Sets

What makes a real number complicated?

The idea: long descriptions = high complexity

Kolmogorov complexity

A string $\sigma \in 2^{<\omega}$ has Kolmogorov complexity $K(\sigma) = n$ if

$$n = \text{length of the shortest program } P \text{ which outputs } \sigma.$$

- $K(\sigma) \leq$ length of σ (almost)
- 0^{1000} has low Kolmogorov complexity.
- Any initial segment of π has low Kolmogorov complexity.
- Random strings exist (by a counting argument).

Notation

For $x = 0.123456789\ldots$ define $x\restriction_5 = 0.12345$ and $x\restriction_n$ similarly for any $x \in \mathbb{R}$, $n < \omega$.

Characterising Hausdorff Dimension

Theorem (Lutz and Lutz, 2018)

If $A \subseteq \mathbb{R}^2$ *then*

$$\dim_H(A) = \min_{Z \in 2^\omega} \sup_{x \in A} \liminf_{n \to \infty} \frac{K^Z(x\restriction_n)}{n}.$$

From (the complexity of) points one can measure the complexity of sets—hence it's called the point-to-set principle.

Lemma

Every countable set has Hausdorff dimension 0.

Proof.

Suppose $A = \{x_i \mid i < \omega\}$. Let $Z = \bigoplus_i x_i$. Let P compute $x_i\restriction_n$ on input (i, n). For fixed i, the pair (i, n) has a description of length $\log(n) + c$, which vanishes $/n$ as $n \to \infty$. □

The $\underset{\sim}{\mathbf{\Pi}}^1_1$-recursion theorem

Theorem (Erdős, Kunen and Mauldin; A. Miller; Vidnyánszky)

($V{=}L$) If at every step of the recursion there exist arbitrarily $\leq_T$-complex witnesses, the constructed set is $\underset{\sim}{\mathbf{\Pi}}^1_1$.

The idea:

1. Well-order the set of conditions $\{c_\alpha \mid \alpha < \omega_1\}$.
2. If $A_\alpha \subseteq \mathbb{R}$ is a partial solution and c_α is not yet satisfied, show that $\{x \in \mathbb{R} \mid x \text{ satisfies } c_\alpha \text{ and } A \cup \{x\} \text{ is a partial solution}\}$ is cofinal in $\leq_T$.
3. Pick such x_α, and define $A = \{x_\alpha \mid \alpha < \omega_1\}$.

Example

($V{=}L$) There is a $\underset{\sim}{\mathbf{\Pi}}^1_1$ decomposition of $\mathbb{R}^3$ into disjoint circles.

The proof

Theorem (R.)

($V{=}L$) *There exists a* $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which* $\dim_H(E) = 1$ *yet for every* θ *we have* $\dim_H(\mathrm{proj}_\theta(E)) = 0$*.*

Proof sketch.

1. Conditions are straight lines through the origin, $\{L_\alpha \mid \alpha < \omega_1\}$.
2. At stage α, find a candidate $x_\alpha \in \mathbb{R}^2$ which has
   - high Kolmogorov complexity by itself, but
   - small Kolmogorov complexity when projected onto L_β for all $\beta \leq \alpha$.
3. Show this set of candidates is cofinal in $\leq_T$. Pick some x_α.
4. (Build x_α by recursion, too, essentially by Cohen forcing.)
5. By $\underset{\sim}{\boldsymbol{\Pi}}^1_1$-recursion, the set $\{x_\alpha \mid \alpha < \omega_1\}$ is $\underset{\sim}{\boldsymbol{\Pi}}^1_1$; by the point-to-set principle, it has the desired properties. □

By enumerating over suitable oracles (and more coding) one can even show the following:

Theorem (R.)

($V{=}L$) *For every* $\epsilon \in (0,1)$ *there exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which* $\dim_H(E) = 1 + \epsilon$ *yet for every* θ *we have* $\dim_H(\text{proj}_\theta(E)) = \epsilon$.

This is optimal by classical facts of geometric measure theory (e.g. Hausdorff dimension cannot drop by more than 1 under projection).

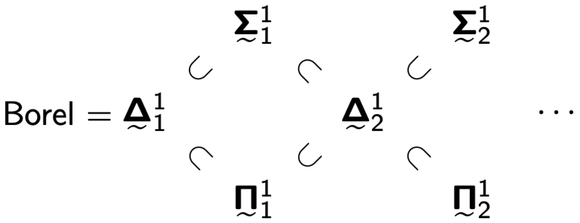
One final example: Erdős-Volkmann Ring Problem

Theorem (Edgar-Miller-Bourgain)

Suppose $R \subseteq (\mathbb{R}, +, \times)$ *is a proper subring. If* R *is* $\undertilde{\mathbf{\Sigma}}^1_1$ *then either* $\dim_H(R) = 0$ *or* $R = \mathbb{R}$.

Theorem (Davies-Mauldin)

(CH) *For every* $s \in (0, 1)$, *there exists a proper subring* $R \subseteq \mathbb{R}$ *such that* $\dim_H(R) = s$. *Under* $(V{=}L)$, *this subring is* $\undertilde{\mathbf{\Sigma}}^1_2$.



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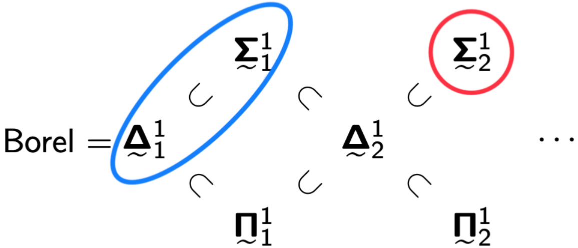

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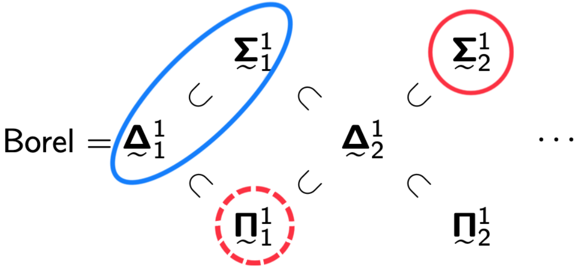

Conclusions

An example of multiple ideas of definability and complexity coming together:

- set theory $\longleftrightarrow$ regularity properties
- to characterise them—and other objects in classical mathematics—use computability theory
  - locally: point-to-set principle for Hausdorff dimension, packing dimension; continuity for functions between polish spaces
  - globally: placement of objects in hierarchies, e.g. Borel/projective hierarchy, arithmetic hierarchy, to prove provability
- many other examples beyond descriptive set theory: e.g. reverse mathematics, computable structure theory

Thank you

Computability in Analysis

A Cauchy name for $x \in \mathbb{R}$ is a sequence $(x_i)_i$ of rationals which converges quickly: $(\forall n)(\forall m \geq n)(|x_m - x| < 2^{-n})$.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there exists P such that: whenever $(x_i)_i$ is a Cauchy name of $x \in \mathbb{R}$ then

$$P^{(x_i)_i}(n) = q_n \in \mathbb{Q} \quad \text{such that} \quad |q_n - f(x)| < 2^{-n}.$$

Theorem

Every computable function is continuous.

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Theorem

Every computable function is continuous.
Every continuous function is computable relative to some oracle.

Computability in Topology

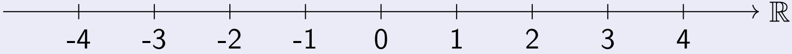
Theorem

A set $A \subseteq \mathbb{R}$ is $\utilde{\mathbf{\Sigma}}^0_{1+\alpha}$ iff there exists P and an oracle Z such that for all $x \in \mathbb{R}$ we have: $x \in A \iff P^{(x \oplus Z)^{(\alpha)}}$ halts in finite time.

Proof by picture of case $\alpha = 0$ (omit oracle).

Want: A is open iff there exists P such that $x \in A \iff P^x$ halts.

($\Leftarrow$):



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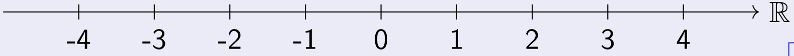

□

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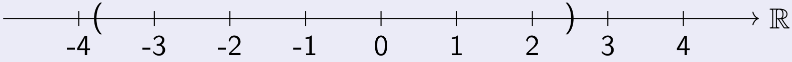

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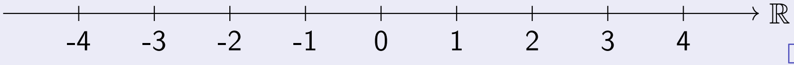

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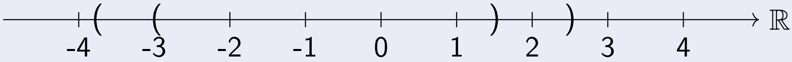

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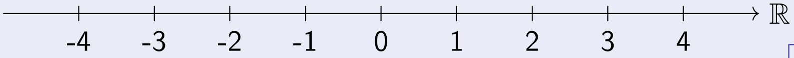

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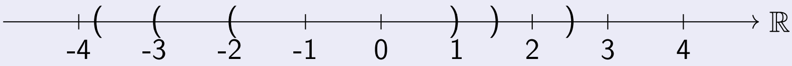

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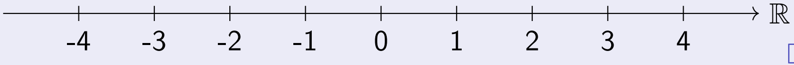

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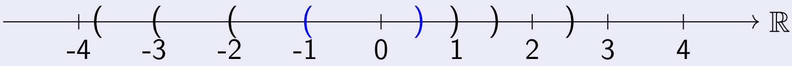

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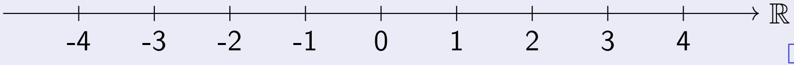

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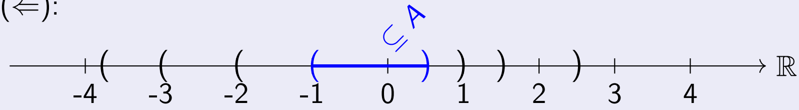

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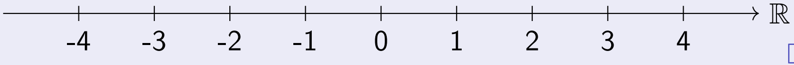

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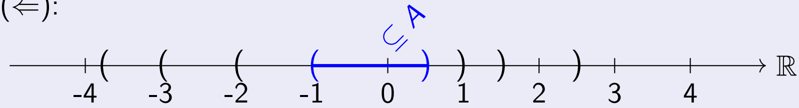

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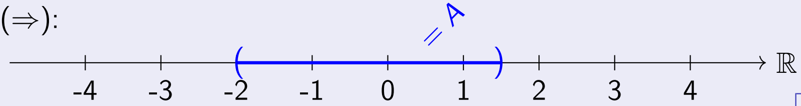

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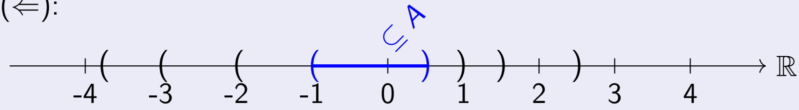

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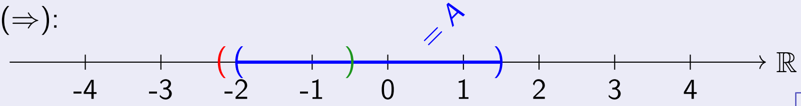

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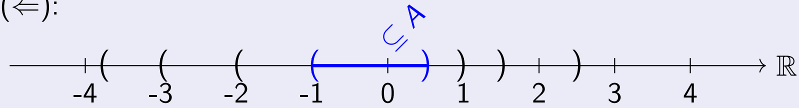

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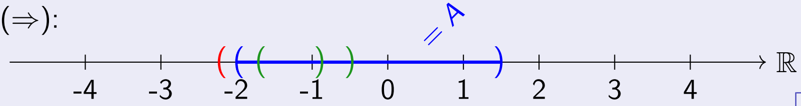

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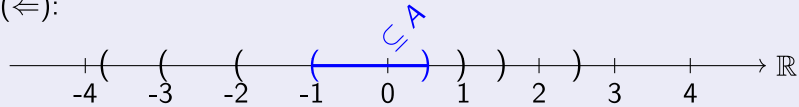

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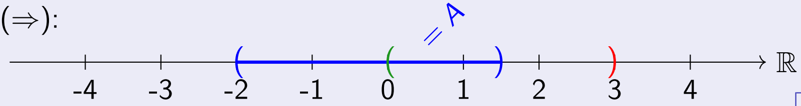

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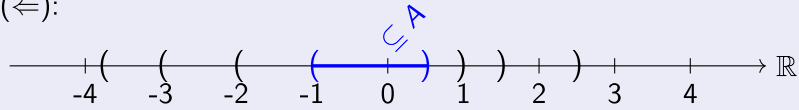

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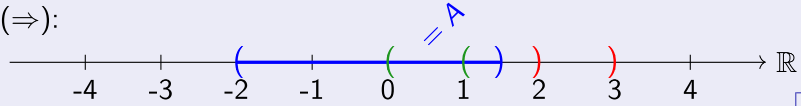

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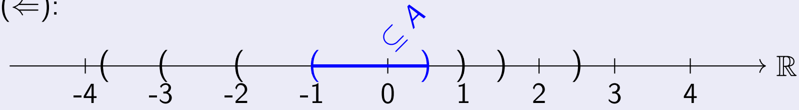

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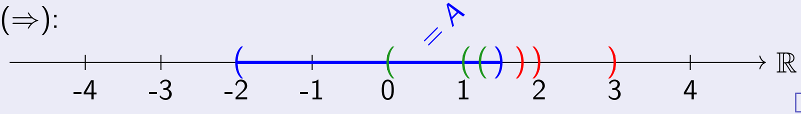

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