Projective Pathways
Towards Roitman's Model Hypothesis

Linus Richter

National University of Singapore

3 September, 2025

Set Theory and Topology in Messina

Question

When will it next snow in Messina?

Question

When will it next snow in Messina?

Ideally, we could find out without waiting for snow, but still in finite time; the problem is computable.

If it's not computable, we'd have to wait *forever* to find out...

Question

When will it next snow in Messina?

Ideally, we could find out without waiting for snow, but still in finite time; the problem is computable.

If it's not computable, we'd have to wait *forever* to find out...

Key insight: ask an oracle!

Definition

A real $A \in 2^\omega$ computes $B \in 2^\omega$ ($B \leq_T A$) if there exists a program which can determine membership of B from finitely many questions to A.

Key property: the use-principle. Computations stop in finite time!

Famously, reals encode information about arithmetic (MRDP-theorem), but they can code much more (there's a whole field dedicated to what reals can code in computable structure theory)!

Most universally, reals code sets:

Lemma (Sacks)

Every set $a \in H(\omega_1)$ can be coded by a real $x \in 2^\omega$.

Set-theoretical Structures in Topology

Roitman's Model Hypothesis is an axiom due to J. Roitman (2011) to settle variants of the box product problem (is $\mathbb{R}^\omega$ under the box topology normal?).

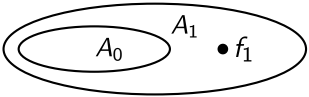
Paul. E. Cohen's Pathways (1979) are a sequence of sets of reals, whose existence implies the existence of P-points.

Recently, Barriga-Acosta, Brian, and Dow related these two.

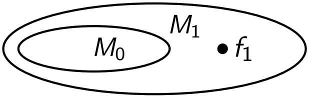
Paul E. Cohen's Pathways:

Roitman's Models:

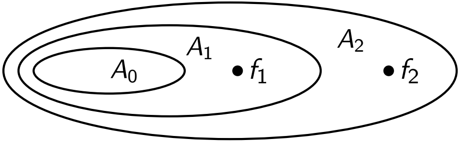
Paul E. Cohen's Pathways:



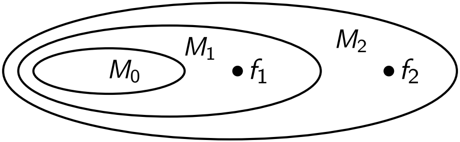
Roitman's Models:



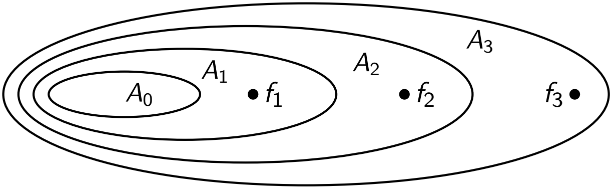
Paul E. Cohen's Pathways:



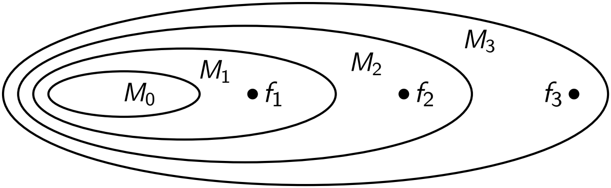
Roitman's Models:



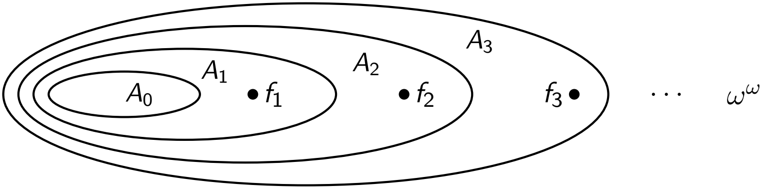
Paul E. Cohen's Pathways:



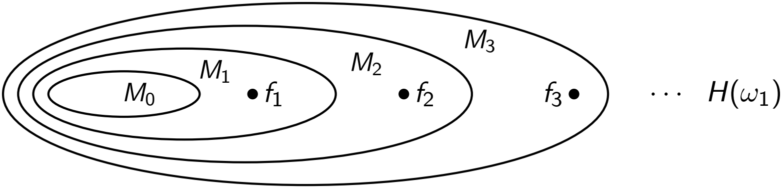
Roitman's Models:



Paul E. Cohen's Pathways:



Roitman's Models:



The fundamental sequences grow more and more complicated!

Definition (P. E. Cohen's Pathways PE)

There exists a cardinal κ and an increasing sequence of sets $(A_\alpha)_{\alpha<\kappa}$ such that:

- $A_\alpha \subset \omega^\omega$
- $\bigcup_{\alpha<\kappa} A_\alpha = \omega^\omega$
- for every α, there exists $f \in A_{\alpha+1}$ such that if $g \in A_\alpha$ then $f \not<^* g$
- A_α is a Turing ideal

Call the sequence $(f_{\alpha+1})_{\alpha<\kappa}$ the fundamental sequence.
The fundamental sequence traces the structure ω^ω.

Definition (Roitman's Model Hypothesis MH)

There exists a cardinal κ and an increasing sequence of sets $(A_\alpha)_{\alpha<\kappa}$ such that:

- $A_\alpha \subset$ ~~ω^ω~~ $H(\omega_1)$
- $\bigcup_{\alpha<\kappa} A_\alpha =$ ~~ω^ω~~ $H(\omega_1)$
- for every α, there exists $f \in A_{\alpha+1} \cap \omega^\omega$ such that if $g \in A_\alpha \cap \omega^\omega$ then $f \not<^* g$
- A_α is ~~a Turing ideal~~ an elementary substructure of $H(\omega_1)$

Call the sequence $(f_{\alpha+1})_{\alpha<\kappa}$ the fundamental sequence.
The fundamental sequence traces the structure ~~ω^ω~~ $H(\omega_1)$.

Definition (Roitman's Model Hypothesis MH)

There exists a cardinal κ and an increasing sequence of sets $(A_\alpha)_{\alpha<\kappa}$ such that:

- $M_\alpha \subset$ ~~ω^ω~~ $H(\omega_1)$
- $\bigcup_{\alpha<\kappa} M_\alpha =$ ~~ω^ω~~ $H(\omega_1)$
- for every α, there exists $f \in M_{\alpha+1} \cap \omega^\omega$ such that if $g \in M_\alpha \cap \omega^\omega$ then $f \not<^* g$
- M_α is ~~a Turing ideal~~ an elementary substructure of $H(\omega_1)$

Call the sequence $(f_{\alpha+1})_{\alpha<\kappa}$ the fundamental sequence.
The fundamental sequence traces the structure ~~ω^ω~~ $H(\omega_1)$.

From Models to Reals

Theorem (Barriga-Acosta, Brian, Dow)

MH *implies* PE.

They also show:

- MH $\implies$ PE $\implies$ P-points exist, so ZFC $\nvdash$ PE, MH
- Neither MH nor PE is equivalent to "P-points exist".
- There are many ccc forcings which give PE, in a sense *via* MH. (If MH is baked into the forcing, then we get PE.)

Can we go the other way?

Does PE imply MH?

On the face of it, the answer *ought to be* no.

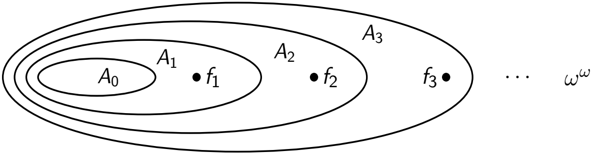
- Pathways are Turing ideals:
  closed downwards under Δ_1 definability.
- Roitman's Models are elementary substructures of $H(\omega_1)$:
  closed under Σ_n definability for all n.

The problem

There's no use-principle for Σ_2, Σ_3, $\Sigma_4, \ldots$ reductions.

By assuming more of our pathways, we can still build models.

Structures Induced by Sets of Reals



Instead of a finite use-principle, we take an "infinite" use-principle via hyperarithmetic reducibility (Kleene):

$$x \leq_h y \iff x \in L_{\omega_1^y}[y] \cap \omega^\omega$$

There is a computability-theoretic interpretation:

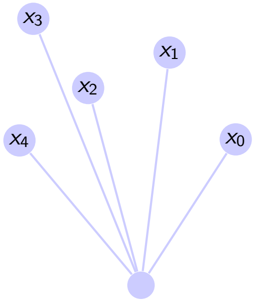
$$x \leq_h y \iff \text{some countable jump of } y \text{ computes } x.$$

So, we capture all Δ_1-, Σ_2-, Σ_3-, $\ldots$ truths and more!

For a set $A \subseteq \omega^\omega$, define

$$L^A := \bigcup_{x \in A} L_{\omega_1^x}[x].$$

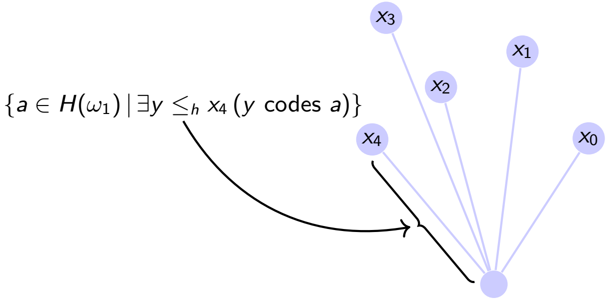
E.g. for $A = \{x_0, x_1, x_2, x_3, x_4, \ldots\}$:



For a set $A \subseteq \omega^\omega$, define

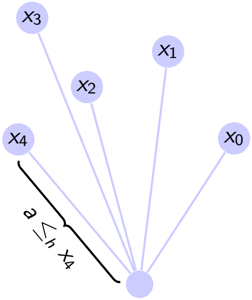
$$L^A := \bigcup_{x \in A} L_{\omega_1^x}[x].$$

E.g. for $A = \{x_0, x_1, x_2, x_3, x_4, \ldots\}$:



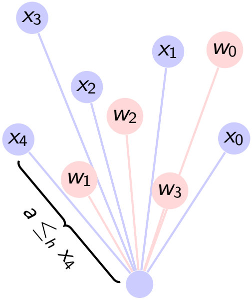
Since $L^A \subset H(\omega_1)$, this is our "induced" structure.

We must extend it to an elementary substructure of $H(\omega_1)$.
$A = \{x_0, x_1, x_2, x_3, x_4, \dots\}$



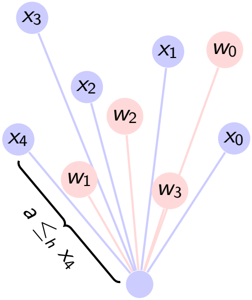
Suppose $H(\omega_1) \vDash \exists x \varphi[x, a]$ for $a \in L^A$.

We must extend it to an elementary substructure of $H(\omega_1)$.
$A = \{x_0, x_1, x_2, x_3, x_4, \dots\}$



Suppose $H(\omega_1) \vDash \exists x \varphi[x, a]$ for $a \in L^A$.
Each w_i codes a witness for φ.

We must extend it to an elementary substructure of $H(\omega_1)$.
$A = \{x_0, x_1, x_2, x_3, x_4, \dots\}$



Suppose $H(\omega_1) \vDash \exists x \varphi[x, a]$ for $a \in L^A$.
Each w_i codes a witness for φ.
The set of witnesses is always projective:

Lemma (Folklore)

$A \subseteq \omega^\omega$ is Σ^1_{n+1} if and only if it is Σ_n over $(H(\omega_1), \in)$.

To guarantee that nice witnesses exist, assume:

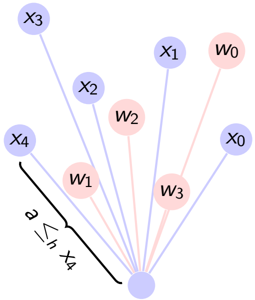
1. A_α is not only a Turing ideal, but a HYP-ideal (i.e. it's closed under $\leq_h$).
2. The fundamental sequence $(f_{\alpha+1})_{\alpha<\kappa}$ grows much more complicated (i.e. it avoids domination by Δ^1_n-reals) .

Call this a $(*)$-pathway.

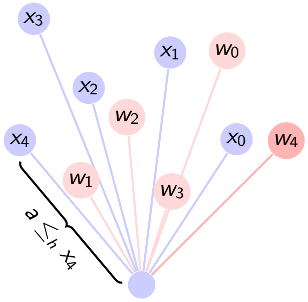
Using a Basis Lemma due to Moschovakis and projective determinacy PD, $(*)$-pathways satisfy:

Lemma

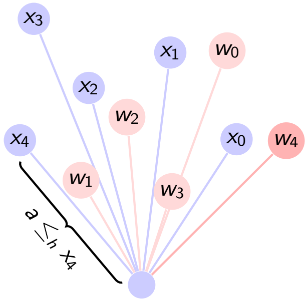
Given L^{A_α}, if $H(\omega_1) \vDash \exists x \varphi[x, a]$, then there is a code for a witness of φ which does not dominate $f_{\alpha+1}$.



x_3
x_1
w_0
x_2
w_2
x_4
x_0
w_1
w_3
$a \leq_h x_4$



(assuming PD and (∗)-pathways)



(assuming PD and $(*)$-pathways)

Theorem (R.)

(PD) *If there is a* $(*)$*-pathway, then* MH *holds.*

Questions

- Can PD be weakened?
- Can closure under HYP be eliminated?
- Can the growth be weakened from “much more complicated” to “more complicated”?

Questions

- Can PD be weakened?
- Can closure under HYP be eliminated?
- Can the growth be weakened from "much more complicated" to "more complicated"?

Thank you