Constructible Failures of the Erdős-Volkmann-Problem for Rings

Linus Richter

National University of Singapore

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Question (Erdős-Volkmann-Ring Problem)

Is there a subring R of $(\mathbb{R}, +, \times, 0, 1)$ such that $\dim_H(R) \neq 0, 1$?

Question

What does this mean?

Descriptive set theory can answer questions about provability.

Fact

Every subset of $\mathbb{R}$ has a Hausdorff dimension.

This dimension can be computed via prefix-free Kolmogorov K complexity and information density: for $x \in 2^\omega$ define

$$\dim(x) = \liminf_{n\to\infty} \frac{K(x\restriction_n)}{n}.$$

J. Lutz and N. Lutz proved the following general identification:

Theorem (J. Lutz, N. Lutz (2018))

For every $A \subseteq \mathbb{R}$ we have

$$\dim_H(A) = \min_{B\in 2^\omega} \sup_{x\in A} \dim^B(x).$$

Definition

An uncountable set $A \subseteq \mathbb{R}$ has the perfect set property (PSP) if it contains a non-empty perfect subset.

Question

Is there an uncountable set $A \subseteq \mathbb{R}$ which does not contain an uncountable subset?

Definition

An uncountable set $A \subseteq \mathbb{R}$ has the perfect set property (PSP) if it contains a non-empty perfect subset.

Question

Is there an uncountable set $A \subseteq \mathbb{R}$ which does not contain an uncountable subset?

Answer

It's complicated.

Axioms	Behaviour
ZF + DC	
ZFC	
ZF + DC + AD	
ZFC + ($V=L$)	

$$\begin{array}{ccccccccc}
 & & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & \\
 & & \subset & & \supset & & \subset & & \\
\text{Borel} = & \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & \cdots \\
 & & \supset & & \subset & & \supset & & \\
 & & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 &
\end{array}$$

Axioms	Behaviour
ZF + DC	PSP holds for all $\utilde{\boldsymbol{\Sigma}}^1_1$ sets (Souslin)
ZFC	
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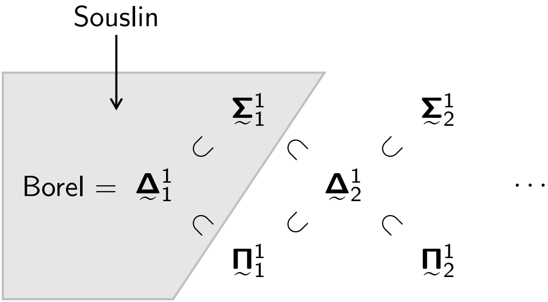
ZFC + (V=L)	



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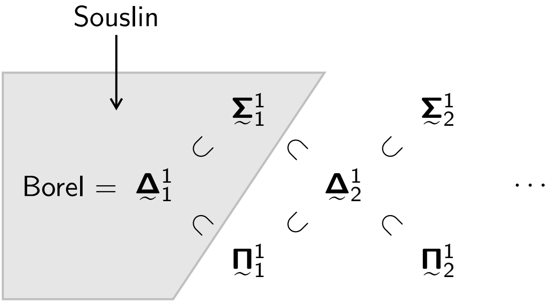

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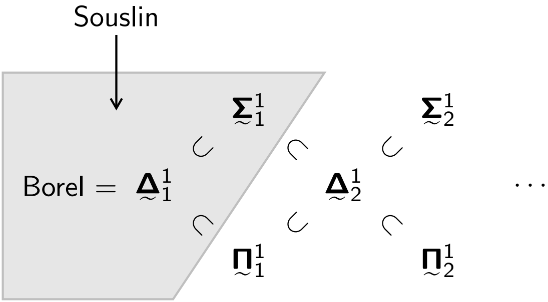

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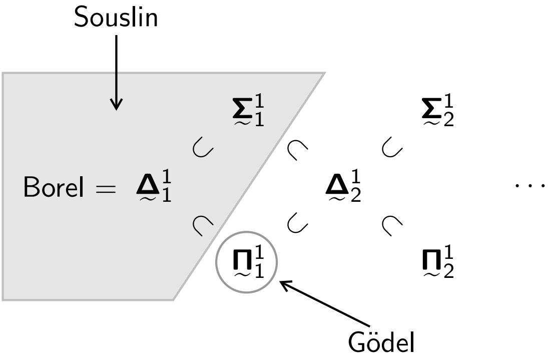
ZFC + ($V = L$)	PSP fails for some $\underset{\sim}{\mathbf{\Pi}}^1_1$ set (Gödel)



Theorem (P. Erdős, B. Volkmann, 1966)

For every $\alpha \in (0,1)$ there exists a Borel subgroup $G \subseteq (\mathbb{R},+)$ for which

$$\dim_H(G) = \alpha.$$

This means, G as a subset of $\mathbb{R}$ is a Borel set.

Question

What about subrings of $\mathbb{R}$?

Theorem (Edgar-Miller, 2001; Bourgain, 2003)

If $R \subseteq (\mathbb{R}, +, \times, 0, 1)$ *is an analytic (i.e.* $\underset{\sim}{\mathbf{\Sigma}}^1_1$*) subring then:*

- *either* $\dim_H(R) = 0$
- *or* $R = \mathbb{R}$*.*

This means, R as a subset of $\mathbb{R}$ is an analytic set.

Theorem (R. D, Mauldin, 2016 (R. O. Davies, 1984))

(CH) *For every* $\alpha \in (0,1)$ *there exists a subring* $R \subseteq (\mathbb{R}, +, \times, 0, 1)$ *such that*

$$\dim_H(R) = \alpha.$$

In fact, R is a subfield. It cannot be $\underset{\sim}{\mathbf{\Sigma}}^1_1$.

Mauldin comments: $V{=}L$ implies the subring can be made $\underset{\sim}{\mathbf{\Sigma}}^1_2$.

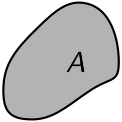
Question

What is the descriptive complexity of such a subring?
Assuming $V{=}L$*, can it be* $\underset{\sim}{\mathbf{\Pi}}^1_1$*?*

Fact

If $A \subseteq \mathbb{R}$ *satisfies* $\dim_H(A) = \alpha$ *then there exists a* G_δ *set* $D \subseteq \mathbb{R}$

$$\dim_H(D) = \alpha \qquad \textit{and} \qquad A \subseteq D.$$



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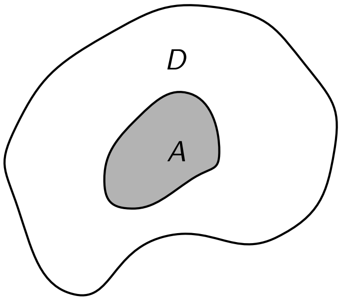

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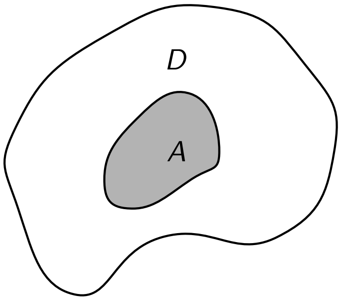

Fact

If A is not contained in any G_δ set of Hausdorff dimension less than α, then $\dim_H(A) \geq \alpha$.

Generalising an idea of Erdős, K. Kunen, and Mauldin (1981), and A. Miller (1989):

Theorem (Z. Vidnyánszky, 2014)

($V{=}L$) Let $P \subseteq 2^\omega$ be uncountable Borel. If $F \subseteq \mathbb{R}^\omega \times 2^\omega \times \mathbb{R}$ is $\underset{\sim}{\mathbf{\Pi}}^1_1$ and for every $(A, p) \in \mathbb{R}^\omega \times 2^\omega$, the section $F_{(A,p)}$ is cofinal in $\leq_T$, then there exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set $R \subseteq \mathbb{R}$ such that

$$P = \{p_\beta \mid \beta < \omega_1\} \quad \text{and} \quad R = \{x_\beta \mid \beta < \omega_1\} \quad \text{and} \quad x_\beta \in F_{(R\restriction_\beta, p_\beta)}.$$

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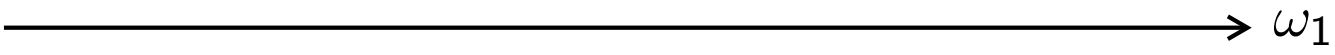
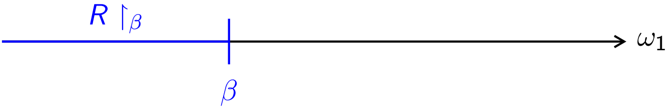
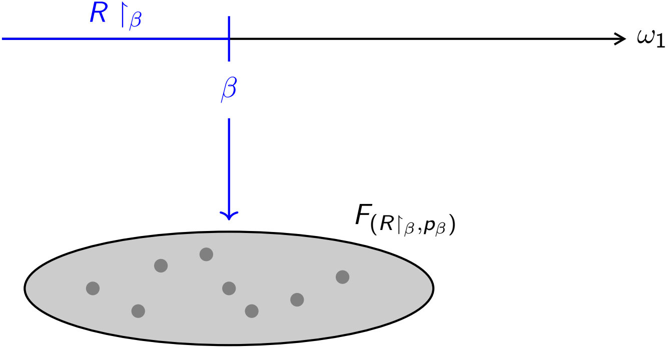
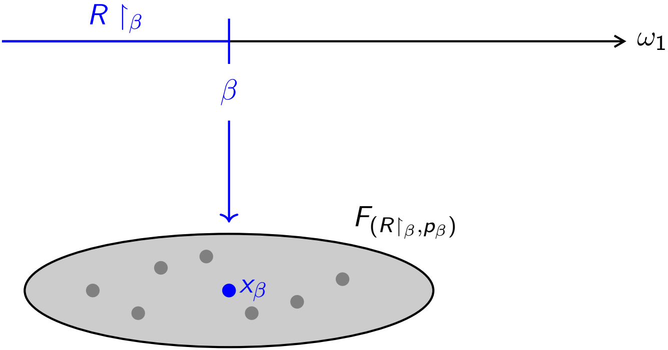

ω_1


R ↾β
β
ω1



$R\restriction_\beta$
ω_1
β
$F_{(R\restriction_\beta, p_\beta)}$



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x_β

Theorem (R.)

For $\alpha \in (0,1)$, *the set*

$$\{c \mid c \textit{ is a } G_\delta \textit{ Borel code for } A \textit{ and } \dim_H(A) \geq \alpha\}$$

is $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$*-complete.*

Corollary

For $\alpha \in (0,1)$, *the set*

$$\{c \mid c \textit{ is a } G_\delta \textit{ Borel code for } A \textit{ and } \dim_H(A) < \alpha\}$$

is $\underset{\sim}{\boldsymbol{\Pi}}^1_1$*-complete.*

Proof (hardness)

Reduce a tree T on ω to a suitable Borel code. The idea:

- A path $x \in [T]$ yields an infinite sequence of nested intervals:

$$\langle 1, 2, 3, \dots \rangle \mapsto \langle (0.1, 0.2), (0.12, 0.13), (0.123, 0.124), \dots \rangle$$

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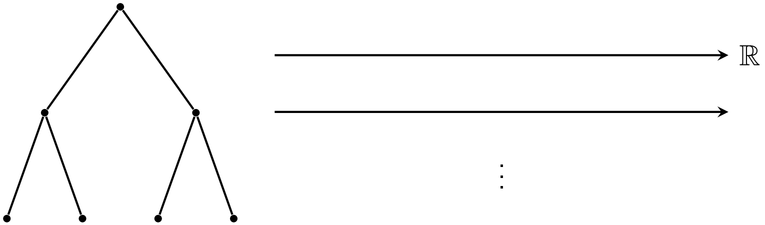

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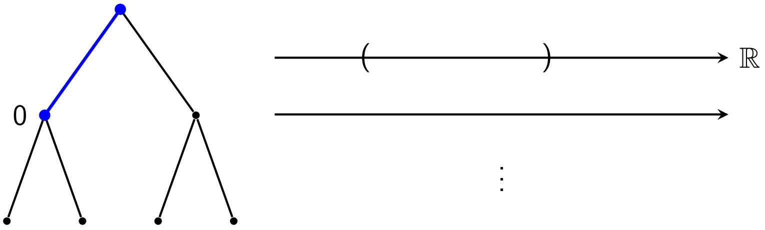

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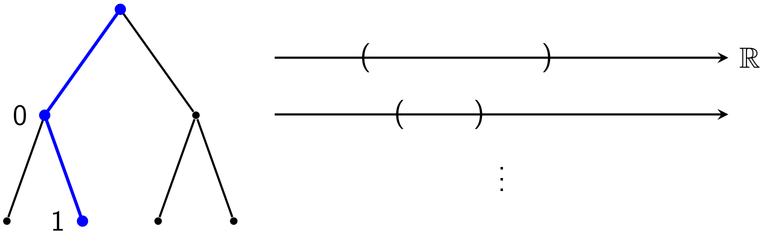
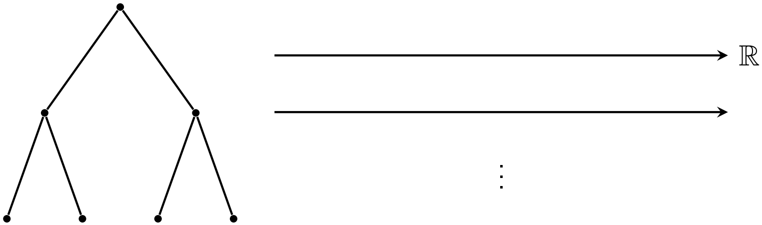


Proposition (R.)

Fix $\alpha \in (0,1)$. Any infinite path $x \in [T]$ codes a G_δ set $A_x \subset \mathbb{R}$ such that $\dim_H(A_x) \geq \alpha$.



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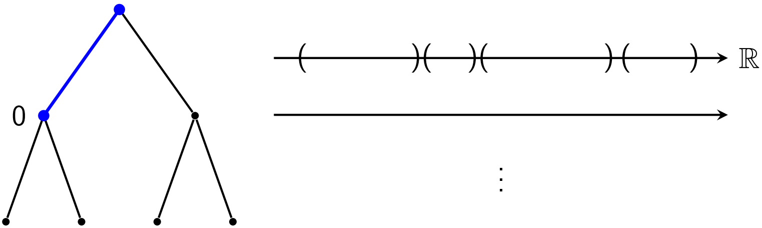

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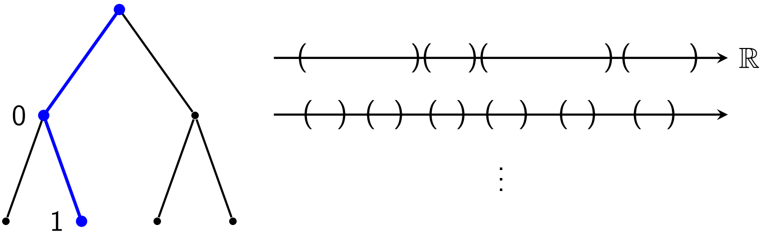

Proof (membership)

Definition

$B \in 2^\omega$ is a Hausdorff oracle for $A \subseteq \mathbb{R}$ if

$$\dim_H(A) = \sup_{x \in A} \dim^B(x).$$

Let A be G_δ. Using the point-to-set principle,

$$\dim_H(A) \geq \alpha \iff (\forall n < \omega)(\exists x \in A)\left(\dim^B(x) > \alpha - 2^{-n}\right)$$

where B is a Hausdorff oracle for A.

Fact

If $A \subseteq \mathbb{R}$ satisfies $\dim_H(A) = \alpha$ then there exists a compact set $K \subseteq A$ such that

$$\dim_H(K) = \alpha \qquad \text{and} \qquad K \subseteq A.$$

Every compact set K is effectively compact relative to some oracle B; then, B is a Hausdorff oracle for K (J. Hitchcock, J. Lutz; D. Stull). Thus,

$$\dim_H(A) \geq \alpha \iff (\exists K \text{ compact})(\forall n < \omega)(\exists B)(\exists x \in K) \Big(K \subseteq A \wedge B \text{ is Hausdorff for } K \wedge \dim^B(x) > \alpha - 2^{-n}\Big).$$

Proposition (R.)

This clause is Σ^1_1 in the Borel code of A.

What about a “direct” construction?

- What are the conditions?
  By the previous theorem, the set of all sufficiently simple G_δ sets is $\underset{\sim}{\mathbf{\Pi}}^1_1$ complete, hence not Borel, hence too complicated.
- Can one build R real by real, by the point-to-set principle?
  This is difficult to apply since we must close under $+$ and $\times$. And since $\mathbb{Q} \subset R$, we cannot make use of compactness.
- Even then, we add countably many reals at every step.
  For every x we enumerate into our subring, we must close under all ring-theoretic operations, which yields (under CH) countably many reals to add. A $\underset{\sim}{\mathbf{\Pi}}^1_1$ basis B gives a $\underset{\sim}{\mathbf{\Sigma}}^1_2$ subring R:

$$x \in R \iff (\exists y \in B)(x \text{ is generated by } y).$$

Question

Is there a subring $R \subseteq (\mathbb{R}, +, \times, 0, 1)$ *with* $\dim_H(R) \neq 0, 1$*?*

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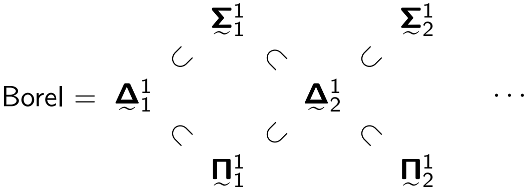

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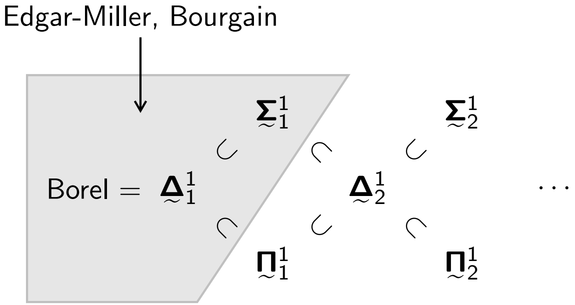

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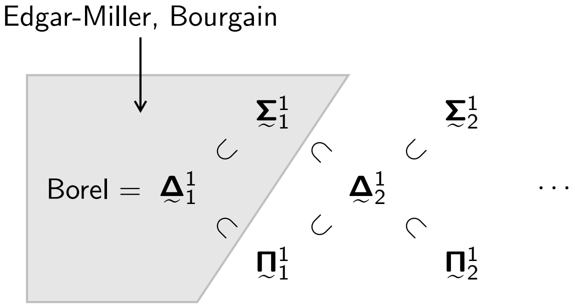

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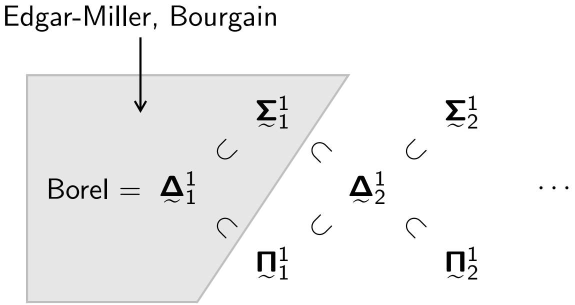

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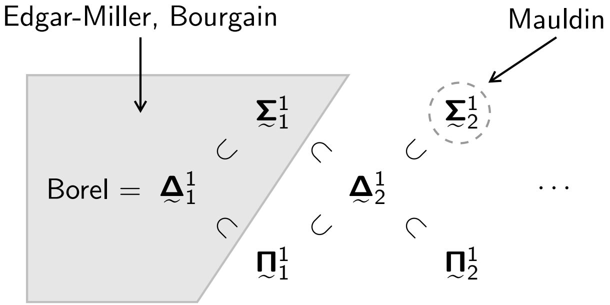

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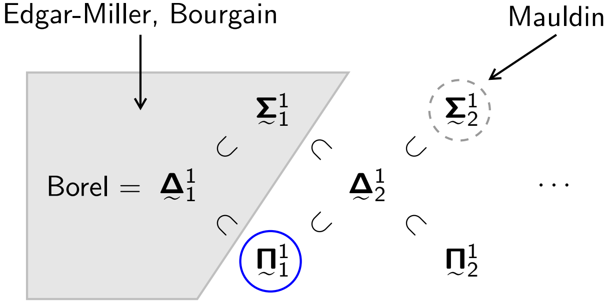

[We] state these results in terms of Borel sets because that is how much we can prove in ZFC, *but they are not really about Borel sets. All of this holds for all constructible sets in* $L(\mathbb{R})$ *if one assumes the large-cardinal hypothesis and for all sets if one assumes [...]* AD *and forgets about the axiom of choice.*

A. Montalbán on Martin's conjecture

NAMS 66(8), 2019

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Thank you