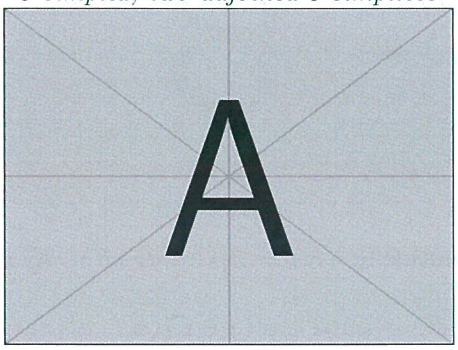
1 Introduction

Martino and I have been working on questions involving Topology, Algebra, and Logic. The connections are somewhat natural in the classical sense. We use algebra to answer topological questions. Fixing objects of a set theoretical nature that are subject to limitations courtesy of ZFC (Baire space, ordinals) allows another angle of approach. Set theoretical tools can now affect the topological properties.

$$\begin{tikzcd} \left[\text{Topology} \arrow[r, bend left] \right. & \left. \text{Algebra} \arrow[l, bend left] \right] & \text{Logic} \arrow[l] \end{tikzcd}$$

We outline an example: consider simplicial homology. This can be likened to graphs: A simplex is a set of points in $\mathbb{R}^n$. A set of $n+1$ points yields n n-dimensional simplex in $\mathbb{R}^n$. A simplicial complex is a set of simplices that is closed under taking intersections and faces. Example:

3-simplex, two adjoined 3-simplices



The dimension of a simplicial complex is the greatest integer k such that the complex does not contain simplices containing k' points.

The set of k-chains is given by linear combinations of k-simplices. So

$$C_r(K) = \{\lambda_i v^i \mid v \text{ is an } r\text{-simplex}, \lambda \in \mathbb{Z}\}$$

so it is the free abelian group generated by the r-simplices. Homology is all about finding nice holes in the simplicial complex. In the standard 3-complex, we expect the interior (i.e. the points that are not on any of the three 2-simplices) to form a hole. The homology groups pick those up. So, here we expect $H_2(K)$ to be isomorphic to the integers (as a guide, the rank of the homology group should give the number of holes).

The boundary homomorphism d^r maps r-chains to $r-1$-chains. In particular, given any r-chain x, its image $d^r(x)$ is an $r-1$-boundary. Chains that are mapped to 0 are called cycles. The r-th homology group is the group of r-cycles modulo r-boundaries. So

$$H_r(K) = Z_r(K)/B_r(K) = \ker(d^r)/\operatorname{im}(d^{r+1})$$

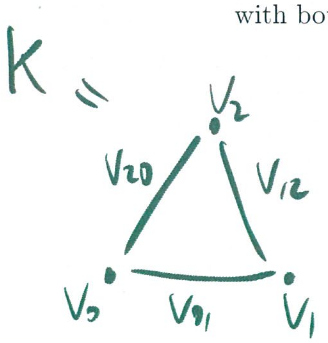
note!

Thus we get a sequence of the following form:

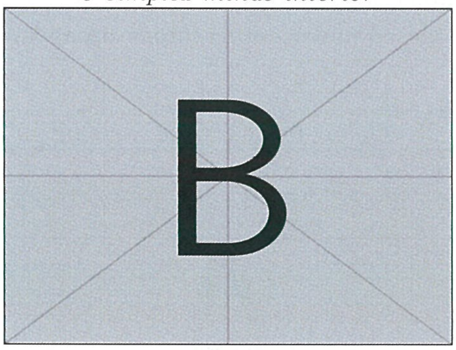
$$0 \leftarrow C_0(K) \xleftarrow{d_1} C_1(K) \xleftarrow{d_2} C_2(K) \leftarrow \dots$$

note!

with boundary homomorphisms connecting the chain complexes.



3-simplex minus interior



Indeed, by definition $H_1(K) = Z_1(K)/B_1(K)$. Again by definition we have that

$$Z_1(K) = \ker(d^1) = \{x \in C_1(K) \mid d^1(x) = 0\}$$

which we can calculate straight away: write v_{ij} in place of $\langle v_i, v_j\rangle$. Then

$$C_1(K) = \{\alpha v_{01} + \beta v_{12} + \gamma v_{20} \mid \alpha, \beta, \gamma \in \mathbb{Z}\}$$

and so cycles are such combinations satisfying

$$d^1(x) = 0 \Leftrightarrow \alpha(v_1 - v_0) + \beta(v_2 - v_1) = \gamma(v_0 - v_2) = 0$$

and solving for parameters we obtain

$$v_0(-\alpha + \gamma) + v_1(\alpha - \beta) + v_2(\beta - \gamma) = 0)$$

<u>Aside</u>: every boundary is a cycle: so

$$B_1(H) \leq Z_1(H).$$

(Combinatorial proof)

Solving this system we obtain that all solutions are given by tuples

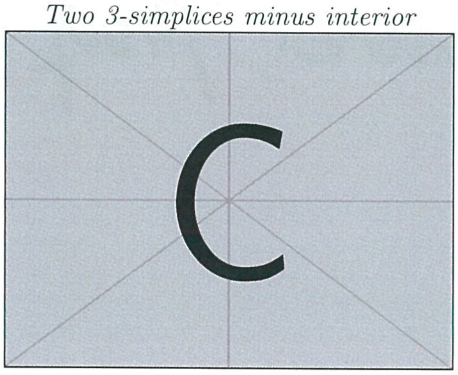
$$(\alpha, \beta, \gamma) = (s, s, s)$$

for $s \in \mathbb{Z}$. Thus

$$Z_1(K) = \{\lambda(v_{01} + v_{12} + v_{20}) \mid \lambda \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Since there are no 2-simplices in K we deduce that $B_1(K) \cong 0$ and thus $H_1(K) \cong Z_1(K) \cong \mathbb{Z}$, as expected.

Two 3-simplices minus interior



$H_1(K) = Z_1(K) / B_1(K).$

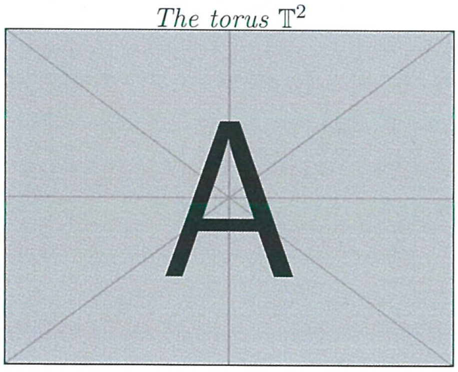
$B_1(K) = \mathrm{im}(d_2) = \{0\}.$

So $Z_1(K)$

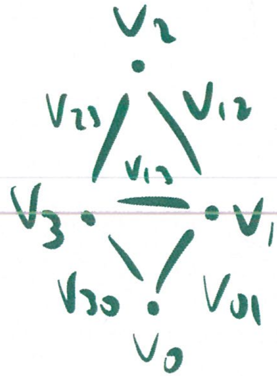
In this case we go through the motions and show that the first homology group is indeed generated by two cycles (not three), and the group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (which underlines our intuition about the rank coinciding with the number of holes).

By this reasoning, we would hope that the torus' simplicial homology groups have rank 2 in dimension 1 and one in dimension 2 (since $\mathbb{T}^2$ has two 1-dimensional holes and one open vacuum).

The torus $\mathbb{T}^2$



K =



$$Z_1(K) = \{ \sigma \in C_1(K) \mid d_1(\sigma) = 0 \} = \ker(d_1).$$

So

$$d_1(\lambda_1 v_{01} + \lambda_2 v_{12} + \lambda_3 v_{23} + \lambda_4 v_{30} + \lambda_5 v_{13}) = 0$$

$$\Leftrightarrow \lambda_1(\cancel{v_1} - \cancel{v_0}) + \lambda_2(\cancel{v_2} - \cancel{v_1}) + \lambda_3(\cancel{v_3} - \cancel{v_4}) + \lambda_4(\cancel{v_0} - \cancel{v_3}) + \lambda_5(\cancel{v_3} - \cancel{v_1}) = 0$$

$$\Leftrightarrow v_0(\lambda_4 - \lambda_1) + v_1(\lambda_1 - \lambda_2 - \lambda_5) + v_3(\lambda_3 - \lambda_4 + \lambda_5) + v_2(\lambda_2 - \lambda_3)$$

So solve :

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So λ_4, λ_5 are free — call s, t.

Then $\lambda_3 = s - t$

$\lambda_2 = s - t$

$\lambda_1 = s$.

So solutions are of the form

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} s \\ s-t \\ s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

So elements of $\mathrm{Ker}(d_1)$ are

$\mathrm{Ker}(d_1) = \{ s\sigma_1 + (s-t)(\sigma_2) + (s-t)\sigma_3$

$+ s\sigma_4 + t\sigma_5 \mid s,t \in \mathbb{Z}\} \cong \mathbb{Z}\times\mathbb{Z}.$

So has rank 2.

... As expected: 2 holes!

A space is triangulable if there is a simplex whose underlying space (as a subspace of $\mathbb{R}^n$) is homeomorphic to it. One can show that the torus is triangulable, and thus use simplicial homology to count its holes.

Homology groups are topological invariants.

2 Cohomology

Cohomology is the dual notion of homology. It allows for more complex algebraic structure (and can hence be considered a stronger or *richer* topological invariant) in the sense that the cup product allows us to combine cocycles of differing dimensions.

When considering the ordinals, this yields insights about the combinatorial properties of ordinals ω_n.

We fix as our space X an ordinal δ.

Suppose $\mathcal{U}_\delta$ is the set of initial segments of δ, that is $\{\alpha \mid \alpha < \delta\}$. This is an open cover of δ. (It can be considered a decomposition of the space just like a triangulation). A presheaf $\mathcal{P}$ associates with each $\alpha \in \mathcal{U}_\delta$ a group of functions from α to a fixed abelian group A. We fix $A = \mathbb{Z}$. The set of r-cochains is now given by functions: we write

$$L^j(\mathcal{U}_\delta, \mathcal{P}) = \prod_{\alpha_0 < \ldots < \alpha_j < \delta} \mathcal{P}(\alpha_0)$$

and thus we obtain a sequence of the form

$$0 \to L^0(\mathcal{U}_\delta, \mathcal{P}) \to L^1(\mathcal{U}_\delta, \mathcal{P}) \to \ldots \to L^j(\mathcal{U}_\delta, \mathcal{P}) \to \ldots$$

connected by the boundary homomorphisms.

There are three important presheaves:

- $\mathcal{D}(\alpha) = \bigoplus_\alpha \mathbb{Z}$;
- $\mathcal{E}(\alpha) = \prod_\alpha \mathbb{Z}$;
- $\mathcal{F}(\alpha) = (\prod_\alpha \mathbb{Z})/(\bigoplus_\alpha \mathbb{Z})$;

DUAL NOTION

For cohomology:

$$0 \to C^0(K) \xrightarrow{d^0} C^1(K) \xrightarrow{d^1} C^2(K) \to \dots$$

and, as before,

$Z^1(K) = \ker(d^1)$ ~ cocycles

$B^1(K) = \operatorname{im}(d^0)$ ~ coboundaries

Usually a stronger topological invariant — cup product allows for ring structure of Cohomology groups (where the cup product acts a multiplication), hence more algebraic tools applicable.

The remainder of this section contains a rather simple example of Borel (even $\mathbf{F}_\sigma$) subgroup of $2^{\mathbb{N}}$ which is not an RN-group. The example is based on a construction, due to Hjorth, which we applied with a similar purpose in mind to other groups in [7, 23].

Let $\mathbb{S} = 2^{<\mathbb{N}}$ be all finite dyadic sequences. For $x \in 2^{\mathbb{N}}$, define $f(x) = \{c_x \restriction m : m \in \mathbb{N}\}$ (this is a branch in $\mathbb{S}$, of course). Let W denote the set of all pairs $\langle x, S\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{S}}$ such that

$$x = x_1 \triangle x_2 \triangle \ldots \triangle x_n \qquad \text{and} \qquad S = f(x_1) \triangle f(x_2) \triangle \ldots \triangle f(x_n),$$

where $x_i \in 2^{\mathbb{N}}$ for all i. It is clear that W is a subgroup of $2^{\mathbb{N}} \times 2^{\mathbb{S}}$, where $2^{\mathbb{N}}$ and $2^{\mathbb{S}}$ are viewed as groups with the symmetric difference as the operation.

For $x \in 2^{\mathbb{N}}$ define $W_x = \{S \in \mathcal{P}(\mathbb{S}) : \langle x, S\rangle \in W\}$. Obviously, $\mathcal{G} = W_\varnothing$ is an $\mathbf{F}_\sigma$ subgroup of the group $2^{\mathbb{N}}$ (not an ideal, of course), while every W_x is a "shift" of $\mathcal{G}$: $W_x = \{S \triangle S' : S' \in \mathcal{G}\}$ for each $S \in W_x$. In addition, we have $f(x) \in W_x$ by definition, so that f is a Borel $\mathcal{G}$-approximate homomorphism.

Lemma 42. *There is no continuous homomorphism $g\colon 2^{\mathbb{N}} \to 2^{\mathbb{S}}$ which $\mathcal{G}$-approximates f. Therefore, $\mathcal{G}$ is not an RN-group.*

Proof. Suppose on the contrary that g is such a homomorphism. Then, by Proposition 39, there is a family of finite sets $u_s \subseteq \mathbb{N}$ (where $s \in \mathbb{S}$) such that, given $x \in 2^{\mathbb{N}}$ and $s \in \mathbb{S}$, $g(x)(s) = 1$ holds if and only if $\#(u_s \cap x)$ is an odd number. (Each $x \in 2^{\mathbb{N}}$ is identified with the set $\{n : x(n) = 1\}$.)

Since g has a value in every set W_x, the set $U = \{s \in \mathbb{S} : u_s \neq \varnothing\}$ cannot be covered by a finite number of branches in $\mathbb{S}$. It follows that there is an infinite set $A \subseteq U$ which is an antichain in $\mathbb{S}$. It is known that in this case there exist an infinite set $A' \subseteq A$ and a finite set v such that $u_s \cap u_{s'} = v$ for any pair of different $s, s' \in A'$. Then obviously there is $x \in 2^{\mathbb{N}}$ such that $\#(u_s \cap x)$ is odd for every $s \in A'$; therefore, $A' \subseteq g(x)$, which contradicts the fact that A' is an infinite antichain. □

11. MEASURABLE COCYCLES AND COHOMOLOGIES

If A, H are abelian (additive) groups, then any map $C\colon A^2 \to H$ satisfying

1*. $C(x,y) = C(y,x)$ and $C(x,y) + C(x+y,z) = C(x,z) + C(x+z,y)$

is called an (abelian) *cocycle* (more exactly, 2-cocycle)[24] of A over H. For instance, given a mapping $\eta\colon A \to H$, the function $C_\eta(x,y) = \eta(x) + \eta(y) - \eta(x+y)$ is a cocycle; cocycles of this kind are called *coboundaries*. Cocycles form an abelian group $Z^2(A,H)$ with the operation $(C_1+C_2)(x,y) = C_1(x,y)+C_2(x,y)$ (for all $x, y \in A$), while coboundaries form its subgroup $B^2(A,H)$. The quotient group $\mathrm{H}^2(A,H) = Z^2(A,H)/B^2(A,H)$ is the *second cohomology group* of A over H.

It is convenient here to take some time for a few simple facts, definitions, and remarks which will be used below. We begin with the following.

2*. We have $C(0,y) = C(0,z)$ for all y, z; indeed, put $x = 0$ in 1*.

If $C\colon A^2 \to H$ is a cocycle then we can define, by induction on $n \geq 2$, a value $C(x_1,\ldots,x_n) \in H$ for any tuple $x_1,\ldots,x_n \in A$ as follows:

3*. $C(x_1,\ldots,x_n,x_{n+1}) = C(x_1,\ldots,x_n) + C(x_1+\ldots+x_n, x_{n+1})$.

Define, in addition, $C(x) = 0$ for all $x \in A$. Then we have

[24] We consider here only this special type of cocycles.

4*. $C(x_1, \ldots, x_n, y_1, \ldots, y_m) = C(x_1, \ldots, x_n) + C(y_1, \ldots, y_m) + C(x, y)$, where $x = x_1 + \ldots + x_n$ and $y = y_1 + \ldots + y_m$.

(Indeed, let, for brevity, $\mathbf{x}$ be $x_1, \ldots, x_n$, and $s = x_1 + \ldots + x_n$. Arguing by induction on m, we apply 1* for $m = 1$. As for the inductive step, assume that

$$C(\mathbf{x}, y_1, \ldots, y_{m-1}) = C(\mathbf{x}) + C(y_1, \ldots, y_{m-1}) + C(s, y_1 + \ldots + y_{m-1}).$$

Adding $C(s + y_1 + \ldots + y_{m-1}, y_m)$, we have $C(\mathbf{x}, y_1, \ldots, y_m)$ on the left, and

$$C(\mathbf{x}) + C(y_1, \ldots, y_{m-1}) + C(y_m, y_1 + \ldots + y_{m-1}) + C(s, y_1 + \ldots + y_m)$$

on the right-hand side by 1*, which is equal to the right-hand side of 4* by 1*.)

Cocycles (including those of much more general form than we consider here) belong to the category of most important algebraic objects. (We refer to the monographs of Cartan and Eilenberg [4, XIV] and Serr [30, VII].) In particular they are used in the theory of group extensions. Indeed, if A, H are abelian groups and $C\colon A^2 \to H$ is a cocycle, then we can define an abelian group, say P_C, with the underlying set $A \times H$ and the operation

goto 35 now!

$$\langle a, h\rangle +_C \langle a', h'\rangle = \langle a + a', h + h' + C(a, a')\rangle,$$

the associativity of which follows from 1*. Now (1) H is a subgroup of P_C (employ the embedding $k \mapsto \langle 0, k - C(0,0)\rangle$), and (2) A is isomorphic to P_C/H (employ the map $a \mapsto \{a\} \times H$), i.e., P_C is an *extension of A with kernel H*. Conversely, any abelian group P satisfying (1) and (2) is presentable as P_C for a map $C\colon A^2 \to H$, and the associativity of the operation implies that C is a cocycle. Further, two extensions, say P_C and $P_{C'}$, are *cohomological* if $\langle A \times H; +_C\rangle$ converts to $\langle A \times H; +_{C'}\rangle$ by a simple shift $\langle a, h\rangle \mapsto \langle a, h + \eta(a)\rangle$, where $\eta\colon A \to H$, which is equivalent to the requirement that $C - C'$ is a coboundary C_η; this leads to the cohomology group $\mathrm{H}^2(A, H)$, of course.

Suppose that the groups A, H considered are Borel (see Section 1). A cocycle $C\colon A^2 \to H$ is called *Borel* if it is Borel as a map. Accordingly, a *Borel coboundary* is any coboundary of the form C_η, where the map $\eta\colon A \to H$ is Borel. Similar definitions are assumed for the other forms of measurability: Baire measurability and the measurability in the sense of a measure μ.

The goal of the following part of the paper is to study measurable, in particular, Borel cocycles and coboundaries. We prove in Section 12 a theorem on relationships between "small" measurable cocycles and coboundaries, which in a sense generalizes the abelian case in Theorem 10. Second, we study the group $\mathrm{H}^2_{\mathrm{Bor}}(A, H)$ of *Borel* cohomologies, i.e., the quotient of the group of all Borel cocycles $C\colon A^2 \to H$ modulo the subgroup of all Borel coboundaries;[25] this group reflects the structure of *Borel* extensions of A with kernel H. We will prove that the group $\mathrm{H}^2_{\mathrm{Bor}}(\mathbb{R}, G)$ is trivial, whenever G is a countable subgroup of the additive group of $\mathbb{R}$, while the group $\mathrm{H}^2_{\mathrm{Bor}}(2^{\mathbb{N}}, 2^{\mathbb{N}})$ is more complicated.

12. "SMALL" COCYCLES AND COBOUNDARIES

Coming back to the case considered in Section 4, let $\mathbb{A}$, μ, $\mathbb{H} = \prod_{n\in\mathbb{N}} H_n$ be as in (*) of Section 4, and let φ be an F-submeasure on $\mathbb{N}$. In addition, we assume that the groups $\mathbb{A}$ and $\mathbb{H}$ (then all H_n) are abelian.

[25] See Moore [29] or Du Pre [10].

BOREL COHOMOLOGY OF $\mathbb{R}^n$ OVER COUNTABLE GROUPS

ML & LR

1. Motivation

Let A, H be abelian groups. We consider maps of the following form:

Definition 1. *A map $C\colon A^2 \to H$ is called an* abelian cocycle *if*

- $C(x, y) = C(y, x)$*;*
- $C(x, y) + C(x + y, z) = C(x, z) + C(x + z, y)$*.*

We recursively define

- $C(x_0, \ldots, x_{n+1}) = C(x_0, \ldots, x_n) + C(x_0 + \ldots + x_n, x_{n+1})$

and set $C(x) = 0$.

The abelian cocycles of the form $C\colon A^2 \to H$ (as defined above) we shall consider below appear prominently in the theory of group extensions.

explain why this follows

Definition 2. *Let G be a group. Then E* extends A by H *if the following is a short exact sequence in the category of groups:*

$$0 \longrightarrow H \longrightarrow E \longrightarrow A \longrightarrow 0$$

In particular, all arrows are group homomorphisms. Then in particular there is a normal subgroup $N \lhd E$ such that $H \cong N$ and $A \cong E/N$.

We also say that E *is an extension of A with kernel H*.

Now let C be an abelian cocycle. Define the group P_C on the ground set $A \times H$ as follows:

$$\langle a, h\rangle +_C \langle a', h'\rangle = \langle a + a', h + h' + C(a, a')\rangle$$

For convenience, define $c_0 = C(0, 0)$. Observe that $c_0 = C(x, 0)$ for any $x \in A$. This holds since $C(x + y, 0) = C(x, 0) + C(x + 0, y) - C(x, y)$.

Lemma 1. *P_C is a group.*

Proof. For associativity, recall $C(x, y) + C(x + y, z) = C(x, z) + C(x + z, y)$. Then $x = \langle a, h\rangle$ and x', x'' defined similarly, we have

$$\begin{aligned}
x +_C (x' +_C x'') &= \langle a, h\rangle +_C \langle a' + a'', h' + h'' + C(a' + a'')\rangle \\
&= \langle a + (a' + a''), h + (h' + h'' + C(a', a'')) + C(a, a' + a'')\rangle \\
&= \langle (a + a') + a'', (h + h' + C(a, a')) + h'' + C(a + a', a'')\rangle \\
&= (x +_C x') +_C x''
\end{aligned}$$

as needed.

The identity in P_C is $\langle 0, -c_0\rangle$ as

$$\langle a, h\rangle +_C \langle 0, -c_0\rangle = \langle a, h - c_0 + C(a, 0)\rangle$$

and since $C(x, 0) = C(y, 0) = c_0$ for all $x, y \in A$.

For inverses, let $\langle a, h\rangle \in P_C$; then its inverse is given by

$$\langle -a, -(h + C(a, -a) + c_0)\rangle$$

Date: May 17, 2021.

We hence obtain

$$\begin{aligned}\langle a,h\rangle +_C \langle -a, -(h+C(a,-a)+c_0)\rangle &= \langle 0, h-(h+C(a,-a)+c_0)+C(a,-a)\rangle \\ &= \langle 0, -c_0\rangle\end{aligned}$$

as required. □

Further, P_C is a group extension of A.

Proposition 1. *P_C is an extension of A by (or with kernel) H.*

Proof. Define an embedding $i\colon H \to P_C$ by $i(h) = \langle 0, h-c_0\rangle$. This is an embedding: it is clearly injective, and a homomorphism since

$$\begin{aligned}i(h) +_C i(h') &= \langle 0, h-c_0\rangle +_C \langle 0, h'-c_0\rangle \\ &= \langle 0, h+h'-2c_0+c_0\rangle \\ &= \langle 0, h+h'-c_0\rangle \\ &= i(h+h')\end{aligned}$$

as needed.

The map $q\colon P_C \to A$ given by $q(\langle a,h\rangle) = a$ is a surjective group homomorphism: it is surjective since $\langle a,0\rangle \in q^{-1}[\{a\}]$. And to see that q, the natural projection, is a homomorphism, observe that

$$\begin{aligned}q(\langle a,h\rangle +_C \langle a',h'\rangle) &= q(\langle a+a', h+h'+C(a,a')) \\ &= a+a' \\ &= q(\langle a,h\rangle) + q(\langle a',h'\rangle)\end{aligned}$$

as needed.

Finally, Observe that $\ker(q) = \operatorname{im}(i)$: for the left-to-right direction, note that $q(\langle a,h\rangle) = 0$ implies that $a = 0$. Then $\langle a,h\rangle = \langle 0,h\rangle = i(h+c_0)$, and hence $\langle a,h\rangle \in \operatorname{im}(i)$. Similarly, $q(i(h)) = q(\langle 0, h-c_0\rangle) = 0$, which proves the reverse inclusion. □

Definition 3. *Let $\eta\colon A \to H$. Cocycles of the form $C_\eta(x,y) = \eta(x)+\eta(y)-\eta(x+y)$ are called* coboundaries*. Two cocycles are called* cohomological *if they differ by a coboundary. So $C \sim C'$ if $C - C' = C_\eta$ for some $\eta\colon A \to H$.*

Definition 4. *Suppose P_C and $P_{C'}$ are extensions of A with kernel H. Then we say that P_C and $P_{C'}$ are* cohomological *if there is an isomorphism $\varphi\colon P_C \to P_{C'}$ such that $\varphi(\langle a,h\rangle) = \langle a, h+\eta(a)\rangle$ for some $\eta\colon A \to H$.*

This indeed makes sense: suppose P_C and $P_{C'}$ are cohomological via η. Then

$$\begin{aligned}\varphi(\langle a,h\rangle +_C \langle a',h'\rangle) &= \varphi(\langle a+a', h+h'+C(a,a')\rangle)) \\ &= \langle a+a', h+h'+C(a,a')+\eta(a+a')\rangle\end{aligned}$$

and similarly

$$\begin{aligned}\varphi(\langle a,h\rangle) +_{C'} \varphi(\langle a',h'\rangle) &= \langle a, h+\eta(a)\rangle +_{C'} \langle a', h'+\eta(a')\rangle \\ &= \langle a+a', h+h'+\eta(a)+\eta(a')+C'(a,a')\rangle\end{aligned}$$

Since φ is a homomorphism, we have $C(a,a') - C'(a,a') = \eta(a)+\eta(a')-\eta(a+a')$. In other words, $C - C' = C_\eta$, a coboundary. Thus chomological group extensions yield cohomological cocycles.

And indeed the converse holds, too: we may deduce that

Lemma 2. *Cohomological cocycles generate equivalent group extensions.*

Proof. We claim that the induced isomorphism $\varphi\colon P_C \to P_{C'}$ with map $\eta\colon A \to H$ makes the following diagram commute:

$$\begin{tikzcd} & & P_C \arrow[dd, "\varphi"] \arrow[dr, "q"] & & \\ 0 \arrow[r] & H \arrow[ur, "i"] \arrow[dr, "i'"'] & & A \arrow[r] & 0 \\ & & P_{C'} \arrow[ur, "q'"'] & & \end{tikzcd}$$

Indeed, recall that $i(x) = \langle 0, x - C(0,0)\rangle$ and $i'(x) = \langle 0, x - C'(0,0)\rangle$, and that q and q' are the canonical projection maps. Then

$$\begin{aligned} \varphi(i(h)) &= \varphi(\langle 0, h - C(0,0)\rangle) \\ &= \langle 0, h - C(0,0) + \eta(0)\rangle. \end{aligned}$$

Now recall that C and C' are cohomological via η, and thus that

$$C(x,y) - C'(x,y) = \eta(x) + \eta(y) - \eta(x+y).$$

Therefore

$$\begin{aligned} C(0,0) - C'(0,0) &= \eta(0) + \eta(0) - \eta(0+0) \\ &= \eta(0) \end{aligned}$$

and so

$$\begin{aligned} i'(h) &= \langle 0, h - C'(0,0)\rangle \\ &= \langle 0, h - (C(0,0) - \eta(0))\rangle \\ &= \varphi(i(h)) \end{aligned}$$

proving commutativity.

The argument for the other diagram is very similar. Hence cohomological cocycles generate cohomological extensions, which in turn are equivalent. □

Thus we may characterise coboundaries by cohomological group extensions P_C of A with kernel H.

Details about this can be found in [Fuc15, 9.1]. The following lemma appears there implicitly.

Lemma 3. *If P is abelian, $H \leq P$ and $A \cong P/H$ then $P = P_C$ for some $C\colon A^2 \to H$.*

Proof. First note that P is abelian, and hence H is normal. And as A is the quotient of abelian groups, it is also abelian. Now consider the short exact sequence

$$0 \longrightarrow H \xrightarrow{\;i\;} P \xrightarrow{\;q\;} A \longrightarrow 0$$

where i is the canonical inclusion map. Define

$$f\colon A \to P$$

so that

$$f(a) \in q^{-1}[\{a\}]$$

(this is sometimes called a *transversal*) and hence let

$$C(a,b) = f(a) + f(b) - f(a+b).$$

We demand that $f(0) = 0$, which we may assume without complications. We confirm that this map C is a cocycle:

- Since A and P are abelian we have

$$\begin{aligned} C(a,b) &= f(a)+f(b)-f(a+b) \\ &= f(b)+f(a)-f(b+a) \\ &= C(b,a) \end{aligned}$$

and so C is commutative in its coefficients.
- Similarly, we have

$$\begin{aligned} C(a,b)+C(a+b,c) &= f(a)+f(b)-f(a+b)+f(a+b)+f(c)-f(a+b+c) \\ &= f(a)+f(b)+f(c)-f(a+b+c) \\ &= f(a)+f(b)+f(c)-f(a+c+b)+f(a+c)-f(a+c) \\ &= C(a,c)+C(a+c,b) \end{aligned}$$

again since A and P are abelian.

In particular, we show that

Claim 1. $\mathrm{im}(C) \subset H$

Proof of Claim 1. By definition of f we have $q(f(a)+f(b)-f(a+b)) = a+b-a-b = 0$, i.e. $C(a,b) \in \ker(q)$. But the sequence above is short exact, so $\ker(q) = \mathrm{im}(i)$. Since i is the canonical inclusion map we have $\mathrm{im}(i) = H$, so we are done. $\dashv$

Therefore $C\colon A^2 \to H$.

We are left to show that $P \cong P_C$ for the cocycle C we have just defined. Recall that P_C is defined on $A \times H$. Hence define

$$\varphi\colon P \to P_C$$

by

$$\varphi(x) = \langle q(x), x - f(q(x))\rangle.$$

Observe that φ is a homomorphism: if $+_C$ denotes addition in P_C then

$$\begin{aligned} \varphi(x+y) &= \langle q(x+y), (x+y) - f(q(x+y))\rangle \\ &= \langle q(x)+q(y), x+y-f(q(x)+q(y))\rangle \\ &= \langle q(x)+q(y), x+y-(f(q(x))+f(q(y))-C(q(x),q(y)))\rangle \\ &= \langle q(x)+q(y), x-f(q(x))+y-f(q(y))+C(q(x),q(y))\rangle \\ &= \langle q(x), x-f(q(x))\rangle +_C \langle q(y), y-f(q(y))\rangle \\ &= \varphi(x) +_C \varphi(y) \end{aligned}$$

by definition of said addition in P_C.

By the Five Lemma, it suffices to show that the diagram

$$\begin{tikzcd} & & P \arrow[dr, "q"] \arrow[dd, "\varphi"] & & \\ 0 \arrow[r] & H \arrow[ur, "i"] \arrow[dr, "i'"'] & & A \arrow[r] & 0 \\ & & P_C \arrow[ur, "q'"'] & & \end{tikzcd}$$

commutes. Then the group extensions P and P_C are in fact equivalent; precisely what we need. Recall that P_C, as a set, is defined on $A \times H$, hence i' and q' are the obvious maps: $i'(h) = \langle 0, h\rangle$ and $q'(\langle a, h\rangle) = a$.

- For the left-hand triangle, note that $\varphi(i(h)) = \varphi(h) = \langle q(h), h - f(q(h))\rangle$. Since $\mathrm{im}(i) = \ker(q)$ we have $q(h) = 0$; we assumed that $f(0) = 0$, and so $\varphi(h) = \langle 0, h\rangle$.
- On the other hand, $q'(\varphi(x)) = q'(\langle q(x), x - f(q(x))\rangle = q(x)$ by definition of q'.

This completes the argument; by the Five Lemma, φ is an isomorphism, and we are done. □

2. Constructing Coboundaries in $\mathbb{R}$

Let C be an abelian Borel cocycle between $\mathbb{R}^2$ and a countable abelian group G. Under which conditions is C a Borel coboundary? Kanovei and Reeken showed in [KR00] the following:

Theorem 1 (Thm 49, Kanovei-Reeken)**.** *If $C\colon \mathbb{R}^2 \to G$ is an abelian cocycle where G is a countable abelian group, then C is in fact a Borel coboundary.*

For $(x_1, \ldots, x_m) \in \mathbb{R}^m$ we write $\mathbf{x}$. We also use standard notation regarding category arguments: By $\forall^* x \in U P(x)$ we mean there is a comeagre set $C \in U$ such that for every $x \in C$ we have $P(x)$. Here is a general argument:

Lemma 4. *Consider $\mathbb{R}^m$ for some $m \geq 1$. Suppose $C\colon \mathbb{R}^m \times \mathbb{R}^m \to G$ is Borel, where $G = \{g_n \mid n < \omega\}$ is countable. There exists an open set $U \subset \mathbb{R}^{2m}$ and $\tilde{g} \in G$ such that if $(\mathbf{x}, \mathbf{y}) \in U$ is generic in U then $C(\mathbf{x}, \mathbf{y}) = \tilde{g}$.*

Proof. View this topologically: the statement asserts the existence of an open set $U \subset \mathbb{R}^{2m}$ and a dense-in-U G_δ-set $Y \subset U$ such that if $(\mathbf{x}, \mathbf{y}) \in Y$ then $C(\mathbf{x}, \mathbf{y}) = \tilde{g}$. We may write

$$\mathbb{R}^{2m} = \bigcup\{X_n \mid n < \omega\}$$

where $X_n = C^{-1}[\{g_n\}]$ since C is total. Also, since C is Borel, there is a dense G_δ-set $Y' \subset \mathbb{R}^{2m}$ on which C is continuous. Put $\tilde{C} = C \restriction Y'$, and hence consider the partition

$$Y' = \bigcup\{\tilde{X}_n \mid n < \omega\}$$

where this time $\tilde{X}_n = \tilde{C}^{-1}[\{g_n\}]$. Now each $\tilde{X}_n$ is closed, by continuity of $\tilde{C}$. Since Y' is a G_δ, it is Polish and thus not a countable union of nowhere dense sets. Thus there exists $n < \omega$ for which $\tilde{X}_n$ contains a non-empty open-in-Y' set, $Y = U' \cap Y'$ say. We may freely assume that U' is basic open in $\mathbb{R}^{2m}$, so $U' = (I_1 \times \ldots \times I_m) \times (J_1 \times \ldots \times J_m)$ where I_i, J_i are open intervals. As Y' is dense in $\mathbb{R}^{2m}$, it is in particular dense in $I_1 \times \ldots \times J_m$, so Y is dense in U' and clearly a G_δ (if $Y' = \bigcap B_n$ where each B_n is open in $\mathbb{R}^{2m}$ then $U' \cap Y' = \bigcap(U' \cap B_n)$). As $Y \subset \tilde{X}_n$, if $(\mathbf{x}, \mathbf{y}) \in Y$ then $C(\mathbf{x}, \mathbf{y}) = g_n$. So $\forall^*(\mathbf{x}, \mathbf{y}) \in U'(C(\mathbf{x}, \mathbf{y}) = g_n)$ as required. □

We may also consider the recursive extensions of C.

Lemma 5. *As defined above, if $C\colon \mathbb{R}^2 \to G$ is Borel then so is $C\colon \mathbb{R}^m \to G$.*

Proof. Of course, we assume G to have the discrete topology. Thus addition in G is continuous. As every continuous map is Borel, and Borel maps are closed under composition, recalling that $C(x_1, \ldots, x_m) = C(x_1, x_2) + C(x_1 + x_2, x_3) + \ldots + C(x_1 + x_2 + \ldots + x_{m-1}, x_m)$ proves the result. □

We aim to control the position of generics in a very precise manner. The Rasiowa-Sikorski lemma allows us to do just that. As always, we assume M is countable.

14. BOREL COHOMOLOGY OF $\mathbb{R}$ OVER COUNTABLE GROUPS

Below, $\mathbb{R}$ will denote the additive group of the reals. This section contains a proof of the following theorem.

Theorem 49. *Let G be a countable abelian group. Then every Borel cocycle $C\colon \mathbb{R}^2 \to G$ is a Borel coboundary; i.e., there exists a Borel function $h\colon \mathbb{R} \to G$ such that $C = C_h$.*[28] *It follows that the group $\mathsf{H}^2_{\mathrm{Bor}}(\mathbb{R}, G)$ is trivial.*

The case $G = \mathbb{Z}$ (integers) in Theorem 49 was proposed by D. Marker as an open problem.[29] Before the proof starts, we present two corollaries.

Theorem 50. *Let $G \subseteq \mathbb{R}$ be a countable group and $f\colon \mathbb{R} \to \mathbb{R}$ be a Borel G-approximate homomorphism.*[30] *Then there exists $r \in \mathbb{R}$ such that $f(x) - rx \in G$ for all x.*

Theorem 51. *Let B be a Borel abelian ordered group having a countable subgroup G as the largest convex subgroup, and B/G be order isomorphic to $\mathbb{R}$. Then B is Borel isomorphic, as an ordered group, to the product $\mathbb{R} \times G$ with the lexicographical order.* $\square$

The derivation of Theorem 51 from Theorem 49 (see [7]) is too complicated to be presented here. On the contrary, Theorem 50, proved in [23], follows easily. Indeed, if a map f is as in Theorem 50, then we apply Theorem 49 to the cocycle $C_f(x, y) = f(x) + f(y) - f(x + y)$. (Clearly C_f is already a coboundary, yet its generating function f maps $\mathbb{R}$ in $\mathbb{R}$ but not in G.) Theorem 49 implies that $C_f = C_h$ for some Borel $h\colon \mathbb{R} \to G$. Then $g(x) = f(x) - h(x)$ is a Borel map $\mathbb{R} \to \mathbb{R}$ satisfying $C_g(x, y) = 0$ for all x; i.e., g is a homomorphism $\mathbb{R} \to \mathbb{R}$. Then g is continuous by the Pettis theorem, which easily implies that $g(x) = rx$ for some r. Finally, $f(x) - g(x) = h(x) \in G$; hence, g G-approximates f.

Proof of Theorem 49. Let us fix a group G and a cocycle C as in Theorem 49. The operation and the neutral element of G will be denoted by $+$ and 0; this will not lead to confusion with the operation and 0 of the additive group of $\mathbb{R}$.

Choose a real $z \in \mathbb{R}$ which effectively codes the Borel map C. Fix a c.t.m. $\mathfrak{M}$ of a big enough finite fragment of **ZFC** which contains z and G. The definition of $\mathfrak{M}[x, y, \ldots]$ and the notion of (Cohen) generic and $\{x, y, \ldots\}$-generic elements with respect to reals and their finite sequences are introduced exactly as in the proof of Theorem 45, assuming that the Cohen forcing is defined as the set of all nonempty rational open intervals (a, b) in $\mathbb{R}$. (Smaller intervals are stronger conditions, as usual.)

By the countability of G, there exist rational intervals I and J and an element $\tilde{g} \in G$ such that I lies at the right of 0 and is shorter than J, and $C(a, b) = \tilde{g}$ holds for each generic (over $\mathfrak{M}$) pair $\langle a, b \rangle \in I \times J$.

Lemma 52. *If reals $x_1, \ldots, x_n, y_1, \ldots, y_n \in I$ are generic and $x_1 + \ldots + x_n = y_1 + \ldots + y_n$, then $C(x_1, \ldots, x_n) = C(y_1, \ldots, y_n)$.*

Recall that the value $C(x_1, \ldots, x_n)$ (here $\in G$) was defined, for any "arity" $n \geq 3$ and for all x_i (here $\in \mathbb{R}$), by 3^* of Section 11.

Proof of Lemma 52. The proof goes by induction on n. We begin with $n = 2$. Let $x, y, x', y' \in I$ be generic and $x + y = x' + y'$; prove that $C(x, y) = C(x', y')$.

[28] Recall that $C_h(x, y) = h(x) + h(y) - h(x + y)$ for all x, y.

[29] Marker, D., private communication, 1998.

[30] This means that $f(x) + f(y) - f(x + y) \in G$ for all $x, y \in \mathbb{R}$.

The proof uses category arguments: one exhibits a subset that contains "many" objects satisfying a specific property. Many can mean: the set has measure 1, the set is comeagre, etc. This is related to forcing: elements in the comeagre set are called generic!

We proved:

$$H^2_{Bor}(\mathbb{R}^n, G) = \{0\}$$

for all $n < \omega$.

It's not clear whether

$$H^2_{Bor}(\mathbb{R}^\omega, G) \text{ or}$$

$$H^2_{Bor}(\mathbb{R}^\omega / \textstyle\bigoplus_\omega \mathbb{R}, G) \text{ is}$$

trivial since these groups aren't Polish.

We suppose that $x < x' < y' < y$. As I is shorter than J, there is a $\{x, x', y, y'\}$-generic real number $\alpha \in J$ such that $\alpha' = \alpha + (x' - x) \in J$. Then each of the pairs $\langle x, \alpha' \rangle$, $\langle y, \alpha \rangle$, $\langle x', \alpha \rangle$, $\langle y', \alpha' \rangle$ is generic by the forcing product theorem. Therefore,

$$C(x, y, \alpha, \alpha') = C(x, \alpha') + C(y, \alpha) + C(x + \alpha', y + \alpha) = 2\tilde{g} + C(\gamma, \gamma'),$$
$$C(x', y', \alpha, \alpha') = C(x', \alpha) + C(y', \alpha') + C(x' + \alpha, y' + \alpha') = 2\tilde{g} + C(\gamma, \gamma')$$

by 4* of Section 11, where $\gamma = x + \alpha' = x' + \alpha$ and $\gamma' = y + \alpha = y' + \alpha'$; hence, $C(x, y, \alpha, \alpha') = C(x', y', \alpha, \alpha')$. On the other hand, still by 4* of Section 11,

$$C(x, y, \alpha, \alpha') = C(x, y) + C(\alpha, \alpha') + C(x + y, \alpha + \alpha'),$$
$$C(x', y', \alpha, \alpha') = C(x', y') + C(\alpha, \alpha') + C(x' + y', \alpha + \alpha'),$$

so that $C(x, y) = C(x', y')$ because $x + y = x' + y'$.

Now, the induction step. Let $x_1 + \ldots + x_n + x_{n+1} = y_1 + \ldots + y_n + y_{n+1}$. First, consider the case $x_{n+1} = y_{n+1}$. Then, $x_1 + \ldots + x_n = y_1 + \ldots + y_n$; hence, $C(x_1, \ldots, x_n) = C(y_1, \ldots, y_n)$ by the inductive hypothesis. On the other hand, by definition,

$$C(x_1, \ldots, x_n, x_{n+1}) = C(x_1, \ldots, x_n) + C(x_1 + \ldots + x_n, x_{n+1}),$$

and the same for $C(y_1, \ldots, y_n, y_{n+1})$, as required.

Now consider the general case. Let x_1 and y_1 be the least, and x_{n+1} and y_{n+1} the largest among the numbers, respectively, x_i, y_i. Let, for instance, $x_1 < y_1$. Take any $\{x_1, y_1, \ldots, x_{n+1}, y_{n+1}\}$-generic real $\varepsilon > 0$ satisfying $\varepsilon < y_1 - x_1$ and such that $y_{n+1} + \delta$ still belongs to I, where $\delta = y_1 - x_1 - \varepsilon$. Define x'_i and y'_i so that

$$x'_1 = x_1 + \varepsilon, \qquad x'_{n+1} = x_{n+1} - \varepsilon, \qquad y'_1 = y_1 - \delta, \qquad y'_{n+1} = y_{n+1} + \delta$$

(these reals are generic by the choice of ε), leaving $x'_k = x_k$ and $y'_k = y_k$ for $2 \le k \le n$. Then $x_2 = x'_2$ and $y'_2 = y_2$, as in the particular case above; hence,

$$C(x_1, \ldots, x_{n+1}) = C(x'_1, \ldots, x'_{n+1}) \qquad \text{and} \qquad C(y_1, \ldots, y_{n+1}) = C(y'_1, \ldots, y'_{n+1}).$$

Similarly, $C(y'_1, \ldots, y'_{n+1}) = C(x'_1, \ldots, x'_{n+1})$, because $y'_1 = x'_1$. $\square$

If $k \in G$ and $m \in \omega$ then let $m \cdot k = \underbrace{k + \ldots + k}_{m \text{ summands}}$ in the group G.

Lemma 53. *Suppose that* $1 \le m < n$, $1 \le m' < n'$, *and generic reals* $x_1, \ldots, x_n, y_1, \ldots, y_m \in I$ *and* $x'_1, \ldots, x'_{n'}, y'_1, \ldots, y'_{m'} \in I$ *satisfy*

$$x_1 + \ldots + x_n = y_1 + \ldots + y_m = s \qquad \text{and} \qquad x'_1 + \ldots + x'_{n'} = y'_1 + \ldots + y'_{m'} = s'.$$

Then

$$(n' - m') \cdot (C(x_1, \ldots, x_n) - C(y_1, \ldots, y_m)) = (n - m) \cdot (C(x'_1, \ldots, x'_{n'}) - C(y'_1, \ldots, y'_{m'})).$$

Proof. If z is a finite sequence of reals (perhaps with only one term), then let $z^{[m]}$ be the sequence of m successive copies of z, say, $\langle x, y \rangle^{[2]} = \langle x, y, x, y \rangle$. Let $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$. Define $\mathbf{x}'$, $\mathbf{y}$, $\mathbf{y}'$ similarly. Then $C(\mathbf{x}^{[n'-m']}, \mathbf{y}'^{[n-m]}) = C(\mathbf{x}'^{[n-m]}, \mathbf{y}^{[n'-m']})$ by Lemma 52. (C applies to strings which consist of $nn' - mm'$ terms and the sum equal to $(n' - m')s + (n - m)s'$.) According to 4^* of Section 11, the left-hand and the right-hand sides of the last equality are equal, respectively, to

$$C(\mathbf{x}^{[n'-m']}) + C(\mathbf{y}'^{[n-m]}) + C\left((n'-m')s, (n-m)s'\right),$$
$$C(\mathbf{x}'^{[n-m]}) + C(\mathbf{y}^{[n'-m']}) + C\left((n-m)s', (n'-m')s\right),$$

so that we have

$$C(\mathbf{x}^{[n'-m']}) + C(\mathbf{y}'^{[n-m]}) = C(\mathbf{x}'^{[n-m]}) + C(\mathbf{y}^{[n'-m']}). \tag{$*$}$$

Using induction on l with 4^* of Section 11, we find that, for any l, $C(\mathbf{x}^{[l]}) = l \cdot C(\mathbf{x}) + C(s^{[l]})$ and $C(\mathbf{y}^{[l]}) = l \cdot C(\mathbf{y}) + C(s^{[l]})$; therefore,

$$C(\mathbf{x}^{[n'-m']}) - C(\mathbf{y}^{[n'-m']}) = (n'-m') \cdot (C(\mathbf{x}) - C(\mathbf{y})).$$

Similarly, $C(\mathbf{x}'^{[n-m]}) - C(\mathbf{y}'^{[n-m]}) = (n-m) \cdot (C(\mathbf{x}') - C(\mathbf{y}'))$. We conclude, by $(*)$, that $(n'-m') \cdot (C(\mathbf{x}) - C(\mathbf{y})) = (n-m) \cdot (C(\mathbf{x}') - C(\mathbf{y}'))$, as required. $\square$

We come back to the proof of Theorem 49. We have to show that C is a Borel coboundary, i.e., $C = C_h = h(x) + h(y) - h(x+y)$ for a suitable Borel "shift" $h\colon \mathbb{R} \to G$. The map h will be a superposition of three more elementary maps.

There exist a big enough natural p and generic reals $x, y \in I$ such that $py = (p+1)x$. An element $\bar g = C(x^{[p+1]}) - C(y^{[p]}) \in G$ (hence, $\in \mathfrak{M}$) satisfies $C(x_1, \dots, x_n) - C(y_1, \dots, y_m) = (n-m) \cdot \bar g$ by Lemma 53 whenever $1 \le m \le n$ and the reals $x_i, y_j \in I$ are generic and satisfy $x_1 + \ldots + x_n = y_1 + \ldots + y_m$.

Step 1. Let $h_1(x) = -\bar g$ and $C_1(x,y) = C(x,y) + C_{h_1}(x,y) = C(x,y) - \bar g$. Since the difference between C and C_1 is a Borel coboundary C_{h_1}, to prove Theorem 49 it suffices to demonstrate that $C_1(x,y)$ also is a Borel coboundary, i.e., $C_1 = C_h$ for a Borel map $h\colon \mathbb{R}^2 \to G$.

Corollary 54. *If generic reals $x_1, \dots, x_n, y_1, \dots, y_m \in I$ satisfy $x_1 + \ldots + x_n = y_1 + \ldots + y_m$ then $C_1(x_1, \dots, x_n) = C_1(y_1, \dots, y_m)$.*

Proof. We have $C_{h_1}(z_1, \dots, z_r) = -(r-1) \cdot \bar g$; hence,

$$C_1(x_1, \dots, x_n) - C_1(y_1, \dots, y_m) = C(x_1, \dots, x_n) - C(y_1, \dots, y_m) - (n-m) \cdot \bar g = 0.$$

(It was supposed that $m \le n$.) $\square$

Recall that the rational interval $I = (a,b)$ lies to the right of 0. Put $nI = (na, nb)$. There is a number $M > b$ such that $[M, +\infty) \subseteq \bigcup_n nI$. Take any $x \ge M$. Then $x = x_1 + \ldots + x_n$ for a suitable string of generic reals $x_1, \dots, x_n \in I$. Put $F(x) = C_1(x_1, \dots, x_n)$; according to Corollary 54, this depends only on x but not on the choice of $x_1, \dots, x_n$. The graph of F is obviously an analytic set; hence, $F\colon [M, +\infty) \to G$ is a Borel function.

Step 2. We put $h_2(x) = F(x)$ for $x \ge M$ and $h_2(x) = 0$ for $x < M$. In particular, $h_2(x) = 0$ for $x \in I$. We note again that, to prove Theorem 49, it now suffices to show that $C_2(x,y) = C_1(x,y) + C_{h_2}(x,y)$ is a Borel coboundary. Note that, by definition, $C_2(x_1, \dots, x_n) = 0$ whenever $x_1, \dots, x_n \in I$ are generic and satisfy $x_1 + \ldots + x_n \ge M$.

Lemma 55. *$C_2(x,y) = 0$ for all $x, y \ge M$.*

Proof. Let $x = x_1 + \ldots + x_n$ and $y = y_1 + \ldots + y_k$, where $x_i, y_j \in I$ are generic. Applying 4^* of Section 11, we have

$$C_2(x_1, \dots, x_n, y_1, \dots, y_k) = C_2(x_1, \dots, x_n) + C_2(y_1, \dots, y_k) + C_2(x,y).$$

Yet $C_2(x_1, \dots, x_n, y_1, \dots, y_k) = C_2(x_1, \dots, x_n) = C_2(y_1, \dots, y_k) = 0$ (see above). $\square$

Step 3. Put $h_3(x) = C_2(x, M_x)$, where $M_x = \max\{M, M - x\}$, so that

$$C_{h_3}(x, y) = C_2(x, M_x) + C_2(y, M_y) - C_2(x + y, M_{x+y}). \qquad (\star)$$

Lemma 56. *$C_2(x, y) = C_{h_3}(x, y)$ for all x, y.*

Proof. We have $C_2(x, y) = C_2(x, z) + C_2(x + z, y) - C_3(x + y, z)$ for every z; hence, the difference $C_2(x, y) - C_{h_3}(x, y)$ transforms, using $(\star)$, to the expression

$$C_2(x, z) + C_2(x + z, y) - C_2(x + y, z) - C_2(x, M_x) - C_2(y, M_y) + C_2(x + y, M_{x+y}).$$

We put $z = \max\{M_x, M_{x+y}, M_y - x\}$. Then, in particular,

$$C_2(x, z) - C_2(x, M_x) = C_2(x + z, M_x) - C_2(x + M_x, z) = 0$$

by Lemma 55. Every one of the other two pairs yields 0 similarly. □

To end the proof of Theorem 49, note that the map h_2 is Borel because so is the map F (see above); thus, the map $h_3 \colon \mathbb{R} \to G$ is Borel as well. It follows that C_2 is a Borel coboundary by Lemma 56, as required.

Theorem 49 is proved. □

ACKNOWLEDGMENTS

The authors thank B. Velickovic, R.I. Grigorchuk, A.S. Kechris, P. Koepke, W. Luxemburg, S. Todorcevic, I. Farah for interesting discussion and/or valuable comments. The first author thanks Universities of Bonn and Wuppertal, Caltech and MFO-Oberwolfach, visiting which in 1999/2000 he fulfilled his part of work over this paper, for support and hospitality.

The first author was supported by grants DFG Wu 101/9-1, DFG 17/108/99, and NSF DMS 96-19880.

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Translated by the authors