VUW LOGIC SEMINAR
CONTROLLING THE CONSTRUCTION OF CA-SETS

LINUS RICHTER

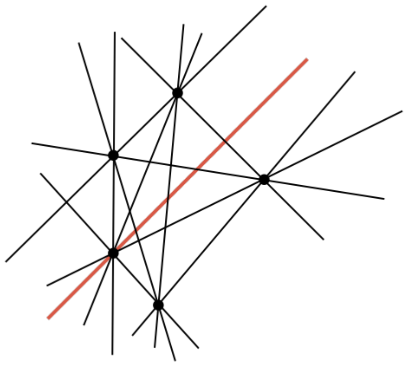
Suppose P is a combinatorial property of sets.

Question: how do we construct a set satisfying P? **Transfinite recursion!**

Example 1. There exists a *two-point set*: a subset of $\mathbb{R}^2$ which intersects every line in exactly two points. Proof [15] and [3, Theorem 6.1.2]: do recursion on all lines $(L_\alpha)_{\alpha<\kappa}$ where $\kappa = |\mathbb{R}|$; choose points at each stage that satisfy the current condition, then later put them all together. Three conditions: at each stage α

(1) choose at most two points;
(2) make sure no three points in our partial solution so far are collinear;
(3) satisfy that there are exactly two points on L_α in our set.

At stage α, suppose P_β is the set of points chosen at stage β for $\beta < \alpha$; let P be the union of all sets P_β for $\beta < \alpha$. Since no three points in P are collinear, $|L_\alpha \cap P| \leq 2$. If it's 2 we are done.



If not, consider the set $\mathcal{L}$ of all lines spanned by any non-degenerate pair of points in P. Then since $|P| \leq 2|\alpha| = |\alpha| < \kappa$, we have $|\mathcal{L}| \leq |P|^2 = |\alpha| < \kappa$ (we overcount by taking *every* pair of points). Each $L \in \mathcal{L}$ intersects L_α at most once (if it hit it twice, then $L = L_\alpha$, so there are already two points of P in L_α); how many points are left? $|L_\alpha \cap \bigcup \mathcal{L}| = |\bigcup_{L\in\mathcal{L}}(L_\alpha \cup L)| \leq |\alpha| < \kappa$. Thus pick one (or two) points from $L_\alpha \setminus \bigcup \mathcal{L}$.

Note 2. Recursion has length $2^{\aleph_0}$, so *probably* not Borel. Original proof due to Mazurkiewicz (1914) [16] (for a French translation), more results by Chad et al [2].

Can a two-point set be Borel? Asked by Erdős [14], still open! (Can't be Σ^0_2 (or in other words F_σ) by [12]; also, if it's Σ^1_1 then it's Borel [17, Section 7].)

Example 3. There exists a partition of $\mathbb{R}^3$ into disjoint circles. Proof [3, Theorem 6.1.3]: well-order $\mathbb{R}^3$ and argue by induction: well-order $\mathbb{R}^3 = \{x_\alpha \mid \alpha < \kappa\}$ for $\kappa = |\mathbb{R}|$; at stage α pick a circle on a plane P_α that doesn't yet contain any previous circle (exists since $|\alpha| < \kappa$ and there are κ many planes containing x_α); previous circles meet P_α in at most two points, so let $S \subset P_\alpha$ be the set of points that lie on a previously enumerated circle in P_α. Take a line $L \subset P_\alpha$ containing x_α, and consider all circles tangent to L that contain x_α (think of the Hawaiian earring). All of these only intersect at x_α, and since

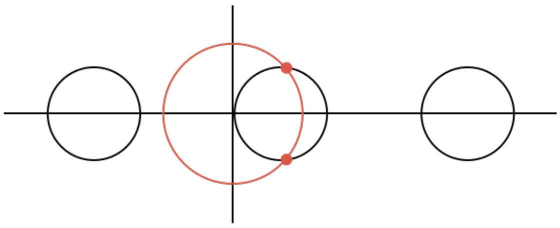
Date: 10 March 2023, Wellington, NZ.

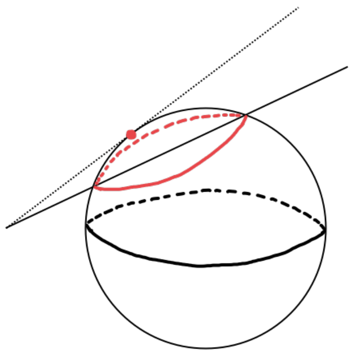
$|S| \leq 2|\alpha| = |\alpha| < \kappa$ (each prev. enum. circle meets P_α in at most two points, and there are $\leq |\alpha|$-many such circles) there must be a circle disjoint from it. Pick that one.

We can do even better: **There exists a partition of $\mathbb{R}^3$ into disjoint unit circles.** Proof: same as above; just note that at stage α each point x that lies in P_α and a previously enumerated circle shares at most two unit circles in P_α with x_α; this disqualifies $4|\alpha| = |\alpha| < \kappa$ many unit circles that contain x_α and lie in P_α; pick any of the remaining unit circles on P_α to complete the proof.

What is the complexity? Tricky question! Is radius fixed? If not, there is a simple construction:

Example 4. Due to Szulkin [21]: place circles on the x-y-plane of unit radius on the points $(4k+1, 0, 0)$ for all $k \in \mathbb{Z}$; let $\mathcal{C}$ be the union of all the circles. S_r is the sphere of radius centred at the origin, then every S_r meets $\mathcal{C}$ in exactly two points (either it's tangent to two distinct circles in $\mathcal{C}$, or it intersects a single circle twice). Consider the sphere S_r minus the two points of intersection with $\mathcal{C}$. Cut it in two halves, each missing a single point, by removing a great circle (a circle through the centre of S_r, so of maximal radius r); each half is an open hemisphere.





The problem now lies in the poles of each hemisphere; it cannot be easily covered by a circle. But each hemisphere misses a point! Hence take the plane tangent to the sphere containing the missing point on the sphere; move that plan across the sphere; each plane intersects with the sphere in a circle; which yields the partition.

Otherwise it's more difficult and might require AC [9]! More thoughts at [5].

Some transfinite recursions can be abbreviated via Zorn's lemma[1]:

Example 5. Considering $\mathbb{R}$ as a vector space over $\mathbb{Q}$, it's an infinite dimensional vector space. **It has a *Hamel basis*.** Proof: take collection $\mathcal{F}$ of all sets that are lin. indep. and only allow a unique representation; this has finite character. Tukey's lemma [7] says a maximal element exists; it does the job.

Example 6. Consider $\mathbb{N}$. **There exists a *MAD family of sets*.** Proof: First build an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$ (e.g. [11]: take for each $f \in 2^{\mathbb{N}}$ the set $S_f = \{\sigma \in 2^{<\mathbb{N}} \mid \sigma \prec f\}$, the set of finite initial segments of f; if $f(i) \neq g(i)$ with $i < \omega$ minimal then $|S_f \cap S_g| = i$, the set of common initial segments of f and g). Then make it maximal via Zorn's lemma, since the union of increasing a.d. families is also an a.d. family.

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[1]the more complicated arguments above don't follow immediately from Zorn's lemma as the maximal element need not satisfy the required conditions

Structure of proofs. Often these proofs are of the following structure:

(1) well-order the set of conditions (lines for two-point set, points for partition of $\mathbb{R}^3$ into circles, etc.)
(2) at stage α: extend partial solution to bigger partial solution *without* breaking the construction, and satisfy condition α
(3) put all parts together

This is called the *diagonalisation technique* [3, Section 6.1]: list all conditions, satisfy each of them one at a time (for cardinality reasons we can always extend: there are only so many lines intersecting a given line, only so many circles in the plane, etc.).

Even the arguments that use Zorn's lemma work this way: partial solutions are glued together to obtain a maximal solutions.

Note 7. A useful observation due to Chad et al [2]: recursions with length $2^{\aleph_0}$ usually ensure that at each stage the set of points satisfying that stage's requirement has *also* cardinality $2^{\aleph_0}$. Hence we actually produce not one set satisfying P but $2^{2^{\aleph_0}}$-many.

If P is such that the collection of sets satisfying P has cardinality $< 2^{2^{\aleph_0}}$ then no such recursion will do the trick; new tools needed!

What's the role of AC? They all use AC in a very obvious manner! In fact, **MAD family requires AC, and Hamel basis requires some choice!** E.g. in [13, Section 5] it's shown that $\mathsf{ZF} + \mathsf{DC}$ has a model without MAD families (assuming large cardinals).

Two-point set: A. Miller showed there are two-point sets in models of ZF where $\mathbb{R}$ is not well-orderable [18]; an earlier construction that deals with all conditions in one step also exists [2].

Szulkin's construction does not require choice.

For Hamel bases: one can construct a model of ZF that doesn't well-order the reals and contains a Hamel basis [1] (in the Cohen-Halpern-Lévy model [20]).

In all of these cases, one can wonder: what is the **complexity of these sets**? The transfinite recursion gives us no handle.

Consider them as subsets of their ambient spaces:

- two-point set: ambient space is $\mathbb{R}^2$
- Hamel basis: ambient space is $\mathbb{R}$
- MAD family: ambient space is $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$
- covering of $\mathbb{R}^3$ for fixed radius: ambient space is $(\mathbb{R}^3)^3$ (just consider centres of circles and two other points on them)

...and all of these are **Polish spaces**! Now we can classify sets with these combinatorial properties using the **Borel** or the **projective hierarchy**! So a natural question follows:

Question 8. What are the "simplest" sets satisfying any of these properties?

Consider the projective hierarchy: Suslin's theorem says: every $\mathbf{\Delta}^1_1$-set is Borel, so that's a useful lower bound. We have the usual inclusions via universal sets:

$$\mathbf{\Delta}^1_1 \subset \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1 \subset \mathbf{\Delta}^2_1 \subset \mathbf{\Sigma}^1_2, \mathbf{\Pi}^1_2 \subset \mathbf{\Delta}^1_3 \subset \dots$$

So, the first step would be to attempt to construct Borel sets with these properties.

Why no Borel sets? Rather difficult: all of the examples above naturally quantify over reals (or second-order objects), which is hard-coded into the definition of analytic sets:

Definition 9. If X is uncountable Polish and $A \subset X$ then $A \in \mathbf{\Sigma}^1_1$ if and only if there exists a closed $F \subset X \times \omega^\omega$ such that

$$x \in A \iff (\exists \alpha)((x, \alpha) \in F)$$

Further, we use a well-ordering in the process of building these sets transfinitely; it is well-known that no Borel well-ordering of the reals exists [6, Lemma 25.41].

In fact it turns out that some of the examples given earlier cannot possibly be Borel! Here are some negative results: there is no $\mathbf{\Sigma}^1_1$

- MAD family [13]
- Hamel basis (there's a proof in [8, Theorem 9]; uses measurability considerations)

Two-point-set (already mentioned)? **Still an open question!**

On the other hand: there exists a $\mathbf{\Pi}^1_1$

- two-point set [17]
- MAD family (ibid.)
- Hamel basis (ibid.)

What do these have in common? **All were shown under $V = L$!**

Question 10. With AC we have: all conditions can be well-ordered, and we can *diagonalise if* the set of "good" candidates is large enough (cardinality-wise). But we have no control over the complexity of the resulting set.

With $V = L$ we have: AC + lots more structure! What do we get in return?

Using $V = L$. Use (effective descriptive) set theory! First approaches due to Erdős, Kunen, and Mauldin [4], and later by A. Miller [17].

Contrast the "classical" case:
AC: suppose M is Polish space.

(1) suppose we build a set $E \subset M$
(2) set of *conditions* $P = \{p_\alpha \,|\, \alpha < \kappa\}$ where $\kappa = |M|$
(3) argue by recursion: at $\alpha < \kappa$, we have only enumerated $< \kappa$ many points **(so there are cardinality-many options left!)**
(4) given partial solution A_α (the choices we've already made) pick $x \in M$ such that:
 (P) $\{x\} \cup A_\alpha$ does not fail the construction[2]
 (D) $\{x\} \cup A_\alpha$ satisfies the condition[3] p_α

Here is an alternative:
$V = L$: suppose M is perfect[4] Polish, computably presented[5].

(1) suppose we build a set $E \subset M$
(2) set of *conditions* P (must be uncountable)
(3) define F so that $(A, p, x) \in F$ iff

EITHER: A is a partial solution and p is not satisfied and x satisfies p and respects[6] A

[2]this often means: don't enumerate too many points
[3]this often means: don't enumerate too few points
[4]so each of its points is a limit point
[5]one needs a recursive Borel isomorphism from M to $2^{\mathbb{N}}$; this is possible if M is computably presented [19, 3I.4]
[6]this means $A \cup \{x\}$ is a partial solution

OR: A is a partial solution and p is satisfied and x respects A
ELSE: A is not a partial solution.

Note 11. A few important notes:
- Any "path" through F constructed by recursion will satisfy all conditions!
- Since $V = L$, every partial solution is countable!
- The set of conditions P is well-orderable!

We never argue by recursion; instead the following theorem does the hard work for us:

Theorem 12 ($V = L$, [22, Theorem 1.3])**.** Let P be uncountable Borel. If F is $\mathbf{\Pi}^1_1$ and each section $F_{(A,p)}$ is *cofinal in the Turing degrees* then the recursion along F yields a $\mathbf{\Pi}^1_1$ set X. By transfinite induction, X satisfies all conditions P.

Note 13. It is now clear why we need the ELSE-case: we must achieve cofinality *for all* pairs (A, p), not just the partial solutions (but of course, those partial solution cases EITHER and OR are the only ones invoked in the construction).

A few words on the proof: only cofinality in hyperdegrees is needed. Builds a set of self-constructible reals[7] $\mathcal{S}$ where

$$\mathcal{S} = \left\{\alpha \,\middle|\, \alpha \in L_{\omega_1^{(\alpha)}}\right\}.$$

Importantly,

$$y \leq_h x \iff y \in L_{\omega_1^{(x)}}[x] \text{ and if } x \in \mathcal{S} \text{ then } L_{\omega_1^{(x)}}[x] = L_{\omega_1^{(x)}}.$$

It is also known [10] that

$$L \cap \omega^\omega = \{\alpha \mid (\exists\beta)(\beta \in \mathcal{S} \wedge \alpha \leq_T \beta)\}.$$

Finally, observe that if $V = L$ and X is $\mathbf{\Pi}^1_1$ then X is cofinal in HYP iff $X \cap \mathcal{S}$ is.

(1) take F as above, suppose $(A, p, x) \in F$;
(2) augment F to some F' of tuples (c, A, p, x) where c codes a well-ordering of conditions which have already been satisfied (still $\mathbf{\Pi}^1_1$; only pick $x \in \mathcal{S}$ s.t. $c \leq_h x$ and $L_{\omega_1^{(x)}}$ agrees that c has coded all conditions so far w.r.t. $\leq_L$, and that p is the next one; possible since all sections are cofinal in Turing degrees)
(3) uniformise to get single "solutions"; not all are, so only keep those tuples which are "paths" through F' (so the history coded by c and A is correct)
(4) pick all x for which there is a partial solution plus history $(c, A, p) \leq_h x$ for which x is the unique solution (still $\mathbf{\Pi}^1_1$ by Spector-Gandy).

Why cofinal in $\leq_h$? If $x \in \mathcal{S}$ then $L_{\omega_1^{(x)}}$ is a true initial segment of L. So if $L_{\omega_1^{(x)}}$ thinks c is a history (of the first α reals w.r.t. $\leq_L$), then it actually is in L, too!

Why $V = L$? Then all reals are constructible, so by the end we will have exhausted all conditions—hence the construction is complete.

Note 14. Is $V = L$ necessary? If Theorem 12 holds then every real is constructible [22, Theorem 4.4]. It's open whether "every real is constructible" suffices [22, Problem 5.7].

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[7] these form the largest thin $\mathbf{\Pi}^1_1$-set, so it's an open question whether any of the constructed sets below can contain perfect subsets [22, Problem 5.8]

Two applications.

Example 15. ($V = L$) **There exists a co-analytic two-point-set.** [22, Theorem 5.2] Conditions are lines coded by reals so that each line appears at least twice.

Let $(A, p, x) \in F$ if and only if

EITHER: there are no three collinear points in A (So A is a partial solution) and the intersection of A with the line p has at most 1 element (So p is not satisfied yet) and $x \in p$ and not collinear with any two points in A (So x satisfies p and respects A)

OR: there are no three collinear points in A (So A is a partial solution) and the intersection of A with the line p has 2 elements (So p is already satisfied) and x is not collinear with any two points in A (So x respects A)

ELSE: there are three collinear points in A (So A is not a partial solution)

This set is Borel; must show each section $F_{(A,p)}$ cofinal in $\leq_T$.

Take a real $r \in \mathbb{R}$. Only interesting cases are the EITHER-OR cases (the ELSE case is obvious; pick any real.

EITHER: we picked $\leq \aleph_0$-many points, so p has $\aleph_1$-many points left (similar as in AC-case); since we can pick one coordinate arbitrarily, that set is cofinal in Turing degrees; it must contain a point that computes r.

OR: just pick any point not breaking the construction (again $\aleph_1$-many).

Hence the theorem applies! Since every line appears at least twice we get a two-point-set that is $\mathbf{\Pi}^1_1$.

Note 16. One can easily build co-analytic n-point sets for any $n < \omega$ by the same argument!

Example 17. ($V = L$) **There exists a co-analytic MAD family.** [22, Theorem 5.1] Conditions are subsets of ω.

The naive approach: iterate over all subsets p; at each stage, if p is almost disjoint from our partial solution, find a superset of p that is also almost disjoint from our solution. **This will work, but won't achieve cofinality in Turing degrees!** Hence:

Let $B = \{B_i \mid i < \omega\}$ be a computable partition of ω into infinite sets.

Let $(A, p, x) \in F$ if and only if

EITHER: $A \cup B$ is a.d. and p is a.d. from $A \cup B$ and $x \supset p$ is a.d. from $A \cup B$

OR: $A \cup B$ is a.d. and p is not a.d. from $A \cup B$ and x respects $A \cup B$

ELSE: $A \cup B$ is not a.d.

This set is Borel; must show each section $F_{(A,p)}$ cofinal in $\leq_T$. Fix $u \subset \omega$.

The idea: code u into finite intersections with B!

EITHER-case: partial solution $A = \{A_i \mid i < \omega\}$, condition p. If $i < \omega$ find finite $F_i \subset B_i$ such that

$$\text{if } j < i \text{ then } A_j \cap F_i = \emptyset$$

and

$$|(p \cup F_i) \cap B_i| \text{ is even } \iff u(i) = 0.$$

Then put

$$x = p \cup \bigcup F_i.$$

Why does this work?

- x is a.d. from B: B is a partition and each $F_i \subset B_i$ is finite; since p is a.d. from each B_i the result follows
- x is a.d. from A: since p is a.d. from A the infinite intersection must be contributed by $\bigcup F_i$; but each A_j only meets finitely many F_i (once $i > j$ then $F_i \cap A_j = \emptyset$), and each F_i is finite
- each F_i exists: $p \cap B_i$ and $A_j \cap B_i$ are finite for each $j < i$; so can pick F_i large enough
- $u \leq_h x$: recall that Theorem 12 holds when sections are cofinal in the hyperdegrees. This is clearly true since[8] $u \leq_T x'$, hence $u \leq_h x$.

The OR-case is very similar, the ELSE-case is trivial: the section is $\mathcal{P}(\omega)$.

Hence Theorem 12 applies! We get a co-analytic MAD family of sets.

References

[1] M. Beriashvili, R. Schindler, L. Wu, and L. Yu. Hamel bases and well-ordering the continuum. *Proc. Amer. Math. Soc.*, 146(8):3565–3573, 2018.

[2] B. Chad, R. Knight, and R. Suabedissen. Set-theoretic constructions of two-point sets. *Fund. Math.*, 203(2):179–189, 2009.

[3] K. Ciesielski. *Set theory for the working mathematician*, volume 39 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[4] P. Erdős, K. Kunen, and R. D. Mauldin. Some additive properties of sets of real numbers. *Fund. Math.*, 113(3):187–199, 1981.

[5] J. D. Hamkins. Concerning proofs from the axiom of choice that $\mathbb{R}^3$ admits surprising geometrical decompositions: Can we prove there is no Borel decomposition? https://mathoverflow.net/q/93601, Aug. 2022.

[6] T. Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[7] T. J. Jech. *The axiom of choice*. Studies in Logic and the Foundations of Mathematics, Vol. 75. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[8] F. B. Jones. Measure and other properties of a Hamel basis. *Bull. Amer. Math. Soc.*, 48:472–481, 1942.

[9] M. Jonsson and J. Wästlund. Partitions of $\mathbf{R}^3$ into curves. *Math. Scand.*, 83(2):192–204, 1998.

[10] A. S. Kechris. The theory of countable analytical sets. *Trans. Amer. Math. Soc.*, 202:259–297, 1975.

[11] K. Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[12] D. G. Larman. A problem of incidence. *J. London Math. Soc.*, 43:407–409, 1968.

[13] A. R. D. Mathias. Happy families. *Ann. Math. Logic*, 12(1):59–111, 1977.

[14] R. D. Mauldin. Problems in topology arising from analysis. In *Open problems in topology*, pages 617–629. North-Holland, Amsterdam, 1990.

[15] S. Mazurkiewicz. Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs. *C.R. Soc. de Varsovie 7*, pages 382–383, 1914.

[16] S. Mazurkiewicz. *Travaux de topologie et ses applications*. PWN—Éditions Scientifiques de Pologne, Warsaw, 1969. Comité de rédaction: K. Borsuk, R. Engelking, B. Knaster, K. Kuratowski, J. Loś, R. Sikorski.

[17] A. W. Miller. Infinite combinatorics and definability. *Ann. Pure Appl. Logic*, 41(2):179–203, 1989.

[18] A. W. Miller. The axiom of choice and two-point sets in the plane. `https://people.math.wisc.edu/~awmille1/res/two-pt.pdf`, 2008.

[19] Y. N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[8] in detail: on input k the jump x' can answer the question "n is the maximal element in $x \cap B_k$" (this is a Π^0_1-question (in x) since each B_k is computable); find the least such n (which exists since $|x \cap B_k|$ is finite); then count the elements that x and B_k have in common below n

[20] D. Pincus and K. Prikry. Lusin sets and well ordering the continuum. *Proc. Amer. Math. Soc.*, 49:429–435, 1975.
[21] A. Szulkin. $\mathbf{R}^3$ is the union of disjoint circles. *Amer. Math. Monthly*, 90(9):640–641, 1983.
[22] Z. Vidnyánszky. Transfinite inductions producing coanalytic sets. *Fund. Math.*, 224(2):155–174, 2014.