ONE WAY TO THINK ABOUT: CHANGE OF BASIS MATRICES

1. Introduction

Suppose $\alpha = \{e_1, e_2\}$ is the standard basis in $\mathbb{R}^2$. We can express any vector $v \in \mathbb{R}^2$ as a *linear combination of vectors in* α: there exist real numbers $x_\alpha, y_\alpha \in \mathbb{R}$ such that

$$x_\alpha e_1 + y_\alpha e_2 = v.$$

The coefficients x_α, y_α are called the *coordinates of* v *with respect to* α.

Of course, we may use a different basis: for instance, consider the basis $\beta = \{g_1 = (1,1), g_2 = (2,1)\}$. It is easily verified that β is a basis; and hence, we can, again express v as a linear combination of vectors in β: there exist reals x_β, y_β such that

$$x_\beta g_1 + y_\beta g_2 = v.$$

Example 1. *Suppose* $v = (2, 3)$*. Then*

$$v = 2e_1 + 3e_2$$

so $x_\alpha = 2$ *and* $y_\alpha = 3$*, and*

$$v = 4g_1 - g_2$$

and hence $x_\beta = 4$ *and* $y_\beta = -1$*.*

This note covers the question: how can we transform (x_α, y_α) into (x_β, y_β)?

2. Always remember the basis

Whenever we consider a "point" in $\mathbb{R}^2$, we implicitly express the point with respect to some basis. Most of the time, this fact stays under the radar as there is a canonical choice for said basis: the standard basis α, mentioned above.

Once we introduce a second basis, β, things get tricky. What is important to remember is that the idea of expressing points with respect to some basis is not new – it is mostly taken for granted.

Consider the example above. We could express the expression of v using the basis α, i.e.

$$v = 2e_1 + 3e_2$$

as the matrix equation

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

which, of course, looks unnecessarily complicated: normally we wouldn't care to write down the identity matrix. But now look what happens when we do the same with the basis β: we obtain

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

and here it is clear why we cannot omit the matrix.

June 13, 2022 (linus.richter@sms.vuw.ac.nz).

3. Now transform

With the matrix equation above in mind, we may write

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

from which we isolate the equation

$$(*) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

This is the equation we always want to keep in mind. What is it saying: it says that the point whose coordinates are $(2, 3)$ with respect to the standard basis α is the same point point whose coordinates are $(4, -1)$ with respect to the basis β!

We will now give these matrices names: let

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_\alpha \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = A_\beta$$

In particular, observe that the columns of A_α are exactly the basis vectors of α, and, similarly, for β. Moreover, the vectors $(2, 3)$ and $(4, -1)$ are, in fact, the coordinates of v with respect to α and β! So we have

$$(\dagger) \qquad A_\alpha \begin{bmatrix} x_\alpha \\ y_\alpha \end{bmatrix} = A_\beta \begin{bmatrix} x_\beta \\ y_\beta \end{bmatrix}$$

which is true in general, for any bases α, β, and any size square matrix (and not just $n = 2$).

To emphasise how useful equation $(*)$ and its more general version $(\dagger)$ above is, we stick with our bases α and β but consider a new point w. Assume we have the coordinates for w with respect to α, let's say they are $(-3, 1)$, but not with respect to β. What can we do? Recall that we have

$$A_\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix} = A_\beta \begin{bmatrix} x_\beta \\ y_\beta \end{bmatrix}$$

and hence

$$A_\beta^{-1} A_\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_\beta \\ y_\beta \end{bmatrix}$$

which allows us to transform the coordinates of w (or any point) from α to β. Hence, we define the *change of basis matrix from α to β by*

$$A_{\alpha\to\beta} = A_\beta^{-1} A_\alpha$$

and, for the other direction, we have

$$A_{\beta\to\alpha} = A_\alpha^{-1} A_\beta$$

as needed.

4. Conclusion

For any two bases α and β, consider the matrix A_α, whose columns are exactly the vectors of α, and similarly A_β, which is constructed from β in the same way. The change of basis matrices $A_{\alpha\to\beta}$ and $A_{\beta\to\alpha}$ are constructed from them via

$$A_{\alpha\to\beta} = A_\beta^{-1} A_\alpha \qquad \text{and} \qquad A_{\beta\to\alpha} = A_\alpha^{-1} A_\beta.$$

If there is one equation to remember for this construction, it is equation $(\dagger)$: it emphasises how every expression of a vector comes with a basis – even though it may be implicit.