On the Definability and Complexity of Sets and Structures

by

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Abstract

We prove results at the intersection of computability theory and set theory, broadly concerning notions of complexity in the sense of definability. We consider these in the contexts of problems in classical mathematics: in the first part of this thesis, we present the solution to a problem in uncountable computable structure theory concerning the construction of complicated uncountable free abelian groups, which was obtained in collaboration with Greenberg, Shelah, and Turetsky. In the second part, we turn towards fractal geometry and its connection with both descriptive set theory and algorithmic randomness. We construct a co-analytic set of reals which fails Marstrand's Projection Theorem, a seminal result of classical fractal geometry and geometric measure theory. The construction uses computability-theoretical tools, in particular the notion of Kolmogorov complexity.

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Chapter 1

Introduction

What does it mean for a mathematical object to be complex? Over the last century, this question has had an immeasurable impact on mathematical logic, and indeed mathematics as a whole. From the complexity of questions and decision problems to the complexity of strings, sets of real numbers, and algebraic structures—measuring what it means for an object to be complicated, and how much information can be coded into it without compromising its inherent order, is a research avenue of great collective interest, not only to logicians but to all mathematicians.

While there are dozens of notions of complexity in mathematics, the most "useful" notion is in the eye of the beholder, and depends on the context at hand. Complex objects could be those which:

- have a very intricate local structure;
- are composed of many smaller parts;
- cannot be decomposed into many smaller parts; or
- cannot be constructed by "simple" operations.

In all of these cases, one must carefully define the terms "intricate", "local", "small", and "simple"—and of course these definitions can differ even within the same context. For instance, consider a null subset of $\mathbb{R}$ whose complement is meagre [127]. John Oxtoby spelled out the ambiguous truth of the smallness of mathematical objects, noting on page 5 in the same source that

> There is of course nothing paradoxical in the fact that a set that is small in one sense may be large in some other sense.

One can replace "small" by "complex", and the statement holds equally true. Regardless of the mathematical formalisation of complexity of objects—of which there are many, and whose choice depends on the context and the problem at hand—the impact of this point of view towards mathematics as a whole is undisputed, and it is our goal to contribute to it with this thesis.

The central question of this thesis is not philosophical—we make no claim as to which definition of mathematical complexity is most useful, let alone correct. Instead, we outline the historical impact of incorporating ideas of complexity into classical mathematics, and contribute to the research stream with two novel results, both at the intersection of classical mathematical subjects with logic. We choose a particular point of view which is one of the fundamental building blocks of the modern subject of mathematical logic.

In this thesis, we study the complexity of mathematical objects via *definability*. A mathematical object X is complex (or complicated) if X is hard to describe: if there is no simple statement defining X in some suitable framework. Both the notion of *simplicity* and that of *definitions* are context-sensitive, as is the surrounding framework. In the classical context of mathematical logic, definitions are formulas, which are expressed in some fixed formal language; and the framework is the model of some fixed theory.

There is the computability-theoretical approach, in which sets are defined by computable operations preceded by potentially infinite searches—these are expressed in terms of unbounded quantifiers; the framework here is classically given by the theory of Turing computation on the set of natural numbers (we give an example in this thesis of a different context). Beyond, there are other, less formal, contexts. There is for instance topology, where sets can be composed via countable operations of open sets—structure is provided by the properties intrinsic to topological spaces; a classical example is descriptive set theory (even though this field admits strong connections to computability theory itself, cf. section 2.1.4). Generally, how to "define definability" depends on the underlying framework, and the notion of *complexity measure*.

The following is a typical example of the type of complexity questions we have in mind as motivation for this thesis. Suppose we would like to measure the complexity of a mathematical object, or of one of its properties. For instance, we might wonder how difficult it is to determine whether a given group is free abelian. An algebraist might suggest this is trivial: a group is free abelian if and only if it has a basis. So, it might be suggested that all we need to do is find a basis. However, this characterisation of free abelian groups does not tell us how complex—or difficult—it is to *determine* whether a given group is free abelian. It does, for instance, not provide us with a plan, an algorithm, a set of instructions, that helps us find, let alone build, a basis. Clearly, more than just the algebraic framework is required to measure the complexity of this difficulty.

Knowing the complexity of characterising free abelian groups is of importance not only for theoretical reasons—well-defined complexity questions can have a direct impact on classical mathematics. Take abelian groups, for example. A group is abelian if and only if its centre is the whole group; this characterisation refers to the centre, a subset of the group. But of course we know that we can define commutativity without referring to subsets: a group with operation $*$ is abelian if and only if for every x and every y, $x * y = y * x$. This is a property of the *elements* of the group, not of *sets* of elements—one can think of this as a *local property.* Putting the classical definition of free abelian groups—having a basis—to one side, an algebraist might wonder whether there is a local property of groups that is sufficient to define free abelian groups *without* reasoning via bases. Perhaps there exists a condition that only refers to group elements—and not subsets of the whole group, such as bases—which, if holding true, implies that the group is free abelian. On the one hand, there is the classical approach to showing the truth of said statement, namely model-theoretically via Gödel's completeness theorem: exhibit a statement T—or, more explicitly, a *formula* in the appropriate language of groups—followed by an argument which proves that if G is a group and G satisfies T then G is free abelian. How would one argue the negative case?

As it turns out, once the appropriate framework is chosen, the negative case can be argued rigorously using tools *outside* the theory (of groups, in this case). We do this by furnishing the ambient complexity framework with *reductions* and a notion of *completeness.* These induce a hierarchy of *complexity measure classes.* Complete representatives are maximally complicated, and any object that can be reduced to them is at most as complicated. Turing degrees are a classical example of such a framework—the Turing degrees are the complexity measure classes, and reductions are given by the 1-reductions.

In our view, the universality of such "reduction/completeness"-frameworks—the fact that they can be adapted and induced in different ambient contexts—is the strongest evidence of the impact of formalised complexity considerations upon classical mathematics.

The main goal of this thesis is to add to the growing list of examples underlining the importance of combining ideas from mathematical logic with classical mathematics. Our contributions include novel applications to *uncountable algebra*—in particular the theory of groups—as well as to *fractal geometry*. Throughout the thesis, we introduce all topics of importance separately. From computable structure theory, descriptive set theory, to fractal geometry and algorithmic randomness; all have their own introductions, which include not only technical details but also historical remarks. In the remainder of this general introduction, we give an overview of the thesis and its structure, and we refer to the relevant dedicated sections, which we invite the reader to examine out of order, if interested.

While definability theory is an immense topic with roots in model theory, we focus on specific notions of definability, which are driven by two underlying frameworks: that of *computability theory* and that of *descriptive set theory*.

In descriptive set theory, the notion of definability—and hence of complexity—is firstly that of Borel measurability. However, particular emphasis deserve those sets which fail to be Borel, the *analytic* and *co-analytic* sets, and those beyond, organised in the *projective hierarchy*. The study of these *definable* sets of reals can be carried out topologically—however, the theory can also be *effectivised*, by investigating it from the lens of classical computability theory. This symbiosis has culminated in the development of *effective descriptive set theory*—which not only subsumes but also extends the classical approach. Our general introduction of descriptive set theory takes place in section 2.1, while effective ideas are being touched upon in section 2.1.4. Co-analytic sets will feature in our work in part II.

Secondly, we consider *higher computability theory*, an analogue of computability theory to uncountable domains based on definability. Its takes place inside the initial levels of the constructible universe L, which we introduce formally. Further, contrary to the study of Borel sets above, it takes some time and care to determine and develop the "correct" measure framework—the context which describes notions of computability theory in uncountable structures faithfully to the classical approach. In this thesis, we base our development of this underlying framework on Noam Greenberg's and Julia Knight's [54]. Set-theoretic background details can be found in section 2.2, while our introduction to higher computability theory follows in chapter 3. It is there where we also give details about the history of *computable structure theory*, the classical arena for complexity questions with respect to Turing computability on ω.

Concerning our novel theorems, we focus on applications of mathematical logic to classical mathematics: *uncountable computable structure theory* in the context of group theory, and *applications of Kolmogorov complexity to fractal geometry.* Each of these are being introduced separately in their respective chapters 3 and 5.

At the core of this thesis we wonder about *definable counterexamples.* Given a property P of some mathematical object (a ring, group, vector space, set of reals, etc.) inside some space which itself admits measures of complexity (by measure, Borel measure, Turing computability, etc.), we ask: what is the *simplest* set failing P? Questions of this type are of great interest as finding definable counterexamples—i.e. objects failing property P of some low complexity in the ambient measure space—informs about the optimality of any existence theorem relating to P. For example: given a set of axioms T, if one can prove in the theory $T + A$ for some additional axiom A that there exists an object failing property P of measure zero, then T cannot prove that property P applies to all null sets. This setup is the motivation to our research, and our work provides examples of this theme.

In part I, we focus on **computable structure theory** in the context of groups of uncountable order. We briefly introduce computable structure theory (cf. [75, 2, 41]). We then develop the theory that allows us to transfer computability theory from the classical domain ω to higher domains in chapter 3—*admissible computability theory*—and base our presentation on expositions of Sacks, and Greenberg and Knight [135, 54, 53]. This development also requires *fine structure* tools which are due to Jensen [64]—we outline these separately in the set-theoretical section 2.2.2; helpful resources are the classical texts of Kunen, Devlin, and Jech [80, 25, 63]. We then turn towards admissibility theory, to prove that L_κ (for a suitable choice of cardinal κ) provides a suitable framework. Set-theoretically, this is explored in the classical way following the work of Barwise [4] and Kripke and Platek [79, 129] on admissibility theory. On the computability-theoretical side, this leads us to α-recursion theory and the work of Sacks [135] and Kreisel [78]—their work was extended by many others in the subsequent decades, under the umbrella of **admissible recursion theory**; among them Maass [97] and Chong [15]. We explore the basics of this subject in section 3.1.2.

We then set up our measure framework on the basis of *definability in L_κ*

in the language of set theory, carefully defining what it means for classes of objects to be **complete** under some suitable notion of **complexity reduction**, and how to produce reductions (cf. section 3.1.3).

We use the developed theory to construct uncountable free abelian groups whose bases are not computable from a fixed oracle. This joint work with Noam Greenberg, Saharon Shelah, and Daniel Turetsky uses the fine structure of L—its combinatorial structure is exploited to build groups which escape computation by a diagonalisation argument. This work answers an open question of Noam Greenberg, Daniel Turetsky, and Linda Brown Westrick which they posed in their work [56]. There, they proved the characterisation of uncountable free abelian groups mentioned at the beginning of this introduction: for most uncountable cardinals κ, there is no first-order definable property that characterises free abelian groups of order κ. Thus, the best characterisation of most uncountable free abelian groups is given by the "obvious" description: the formalised statement "there exists a basis". Formally, they showed that the set of free abelian groups of said order is Σ^1_1-complete; so, given the properties of the underlying framework of complexity measure—in this case admissibility theory—characterisations of free abelian groups cannot be any simpler[1] (see Theorem 3.2.5). From this characterisation it follows that every Δ^1_1 oracle fails to compute the basis of some free abelian group (see Corollary 3.2.6), but prior to our work it was not clear *how* to construct such a witness—this explicit process is described in chapter 4.

In part II, we turn towards **fractal geometry**. We focus on John

[1]The theme of the argument—show that a set is *complete* in some framework which specifies a notion of *reduction*—is a classical approach to proving hardness in logic [154, 155, 72], evidencing the programme's versatility.

Marstrand's projection theorem (cf. Theorem 6.1.1), one of fractal geometry's classical theorems [102]. His result relates the Hausdorff dimension of sets of reals in $\mathbb{R}^2$ to that of its projections onto straight lines through the origin: Marstrand showed that, modulo a null set, the Hausdorff dimension of the projection of any analytic subset of $\mathbb{R}^2$ is maximal. In the late 1960s, Kaufman found a proof of Marstrand's very technical arguments in terms of capacities [70], before, in 1975, Marstrand's results were fully generalised by Pertti Mattila; he showed that the projections of all analytic sets in $\mathbb{R}^n$ behave equally well for all projections onto subspaces of Hausdorff dimension $m < n$ for all m and all n [106].

Interest in projection questions has remained high ever since—especially as a perhaps surprising connection between fractal geometry and a classical measure of complexity of real numbers has been discovered, examined, and well-understood over the last thirty-five years. In the 1980s, Ryabko [132, 133] explored the connection between notions of complexity with Hausdorff dimension, followed by Staiger [146], as well as Cai and Hartmanis [10] in the 1990s, who examined the impact of incorporating *Kolmogorov complexity* theory into questions of Hausdorff dimension. This research stream gained additional traction in the early 2000s with Jack Lutz' work on effective dimension of reals [89]. His work focussed on *elements* of the Cantor space 2^ω (not only on *sets* of reals), and was quickly improved upon by Mayordomo [110], Hitchcock [59, 60], Athreya et al. [3], and others. The former found a representation of Lutz' dimension of reals in terms of information density; hence, we can express Lutz' effective dimension via *Kolmogorov complexity*, a classical tool of computability theory. This work has culminated in a recent identification of Hausdorff dimension in terms of algorithmic complexity of its individual points, the so-called **point-to-set principle**, due to Jack Lutz and Neil Lutz [91], which has proven to be immensely applicable [95, 96, 94, 147]. It has also been extended to spaces beyond 2^ω [92].

In chapter 6—the arguments of which have been submitted for publication [131]—we exhibit a set that fails Marstrand's theorem as badly as possible, under the set-theoretical assumption that "every set is constructible", denoted by $V{=}L$. The significance of our theorem is given by the set's complexity: our set is co-analytic, or $\utilde{\mathbf{\Pi}}^1_1$, in the projective hierarchy. Paired with Marstrand's original theorem, which provably applies to every analytic, or $\utilde{\mathbf{\Sigma}}^1_1$, set of reals, we hence construct optimal counterexamples. The proofs of our theorems are algorithmic, and use at their core the identification of fractal geometry in terms of Kolmogorov complexity: we use the aforementioned point-to-set principle. The descriptive set-theoretical tools we employ are due to Zoltán Vidnyánszky [152], who proved a co-analytic recursion principle whose significance we explain in section 2.3.

1.1 Technical Conventions and Notation

We assume that the universe satisfies ZFC, unless otherwise stated; this is **Zermelo-Fraenkel set theory** including the axiom of choice AC. We mean by ZF the theory of ZFC minus the axiom of choice AC, and by ZF^- we describe ZFC without AC and the axiom $\mathsf{Power\ Set}$. The continuum hypothesis is denoted by CH, and GCH denotes its generalisation. The axiom $V{=}L$ denotes the formalised sentence "every set is constructible", and L denotes Gödel's constructible universe. This list is not exhaustive; in later section we introduce further set-theoretical axioms, such as $\mathsf{Collection}$ in section 3.1.1.

The alephs are denoted by $\aleph_\alpha$ for any ordinal α, but we frequently switch between $\aleph_0$ and ω, describing the same ordinal. Generally, ordered pairs of sets are denoted by $(\cdot,\cdot)$, and the relation $A \subset B$ denotes set inclusion *not* excluding the case $A = B$, unless otherwise stated.

The following sets appear repeatedly throughout this work: ω denotes the set of natural numbers $\{0, 1, 2, 3, \ldots\}$. ω^ω denotes the set of functions from ω to ω, and we similarly define 2^ω. We frequently identify these functions—which we also freely call **reals**—with infinite sequences of natural numbers (in the case ω^ω), or of $0s$ and $1s$ (in the case 2^ω). For any cardinal κ and any set A, we define $A^{<\kappa} = \bigcup\{A^\lambda \mid \lambda < \kappa\}$. These appear in particular as $\omega^{<\omega}$ and $2^{<\omega}$. If $\sigma \in \omega^{<\omega}$, then $[\sigma]$ denotes the set of all $f \in \omega^\omega$ of which σ is an initial segment; this naturally applies to all spaces of the form A^κ for all sets A and all cardinals κ.

If $A \subset X \times Y$, then we define the **projection of A onto X** to be the set $\{x \in X \mid (\exists y \in Y)((x, y \in A))\}$, denoted by $\mathrm{proj}_X(A)$, if the coordinate to be projected upon is clear. Otherwise, we write $\mathrm{proj}_1(A)$, $\mathrm{proj}_2(A)$, and so forth. If we project multiple coordinates, we also write $\mathrm{proj}_{1,2}(A)$.

Chapter 2

Background

We give a brief introduction to two important cornerstones of our work below that feature throughout. First, we introduce the origins and basics of the theory of Polish spaces, and in particular its relationship to complexity considerations: the Borel and projective hierarchies—these are the primary components of *descriptive set theory*. We then mention the connection between the classical and effective hierarchies, which link the classical theory with computability theory—in contemporary work in descriptive set theory this connection is indispensable.

Then, we focus on the *constructible universe*, and in particular on its combinatorial properties. We outline the motivation and relevant results of Jensen's fine structure theory, which will be of use to us in later sections.

Finally, we talk about recursive constructions of sets of reals. We outline the importance of AC, and explain Vidnyánszky's theorem, which proves how to recursively construct co-analytic sets of reals, under the assumption $V{=}L$.

2.1 Descriptive Set Theory

A principal branch of contemporary set theory is the study of well-behaved sets of the real numbers. This theory was born in the early 20th century, following Henri Lebesgue's flawed "proof" that the collection of Borel sets (those constructible from the open sets via countable sequences of countable unions and complementations) is closed under projections [83]. Some say that the subject of **descriptive set theory** was born in 1917 with Mikhail Souslin, then a student of Nikolai Lusin's, who observed Lebesgue's mistake: he noticed that Lebesgue falsely assumed that intersections and projections commute. Rectifying this shortcoming, he constructed a subset of the real line which was provably more complicated than any Borel set, hence showing that not every subset of $\mathbb{R}$ is Borel (in the parlance of the time, he showed there exists an A-set which is not a B-set, a fact whose converse Pavel Aleksandrov had tried to prove for at least a year [1]; that said, the naming of "analytic" sets is somewhat controversial historically [68, p. 148]). This discovery jumpstarted the investigation of Souslin sets, and in extension that of **analytic sets** and of their structure [142]—sets appearing in this hierarchy beyond the Borel sets are nowadays called **projective sets**.[1]

The sets studied in descriptive set theory are often called *definable sets of reals*. This is because their Borel (or projective) complexity can be measured precisely by identifying their level in said hierarchies, and since the hierarchies are constructed in a bottom-up process whose basic building blocks are the open (and closed) sets. As expected, sets in different levels of this hier-

[1]For more historical details on the early history of descriptive set theory, pioneered by the Russian school spearheaded by Nikolai Lusin, see section 12 of [68].

archy satisfy different structural properties—for example, every Borel set is Lebesgue measurable; hence the non-measurable Vitali set cannot be Borel—whose investigations form a cornerstone of descriptive set theory today.

We give a brief technical introduction of the basics of descriptive set theory below, as the measure of complexity of subsets of Polish spaces in terms of the Borel and the projective hierarchy is of importance to us in part II of this thesis. There, we contrast this notion of complexity with those of Hausdorff measure and dimension.

Let X be a separable and completely metrisable topological space—such spaces are called **Polish**. Classical examples of Polish spaces include $\mathbb{R}$, $\mathbb{R}^n$, the Cantor space 2^ω, and the Baire space ω^ω. The subsets of Polish spaces that are constructed from the open sets by countable sequences of operations give rise to a rich hierarchy: the **Borel hierarchy**. We first give an elementary fact about Polish (and in fact all separable metrisable) spaces.

Lemma 2.1.1. *Every Polish space is second countable.*

The open sets contained in a basis are called **basic open**. Certain Polish spaces have canonical bases, e.g. the sets $\{f \in 2^\omega \mid \sigma \text{ is an initial segment of } f\}$ for every $\sigma \in \omega^{<\omega}$ form a basis for 2^ω.[2]

We now turn towards closed sets. Clearly, not every closed set is open (this is obvious for $\mathbb{R}$ but equally true for every uncountable Polish space). However, countable intersections of open sets recover all closed sets:

Lemma 2.1.2. *In every metric (and hence also every Polish) space, every closed set is a countable intersection of open sets.*

[2]Hence 2^ω is zero-dimensional, as is ω^ω by the same argument.

We now give an introduction of the Borel hierarchy of a fixed Polish space X below. Let $\underset{\sim}{\mathbf{\Sigma}}^0_1(X)$ denote the open subsets of X. Fix a countable basis $\{U_n \mid n < \omega\}$ of X. By Lemma 2.1.1 every open set can be written as a countable union of the basic open sets U_n. Thus, if $U \subset X$ is open then $U = U_{n_0} \cup U_{n_1} \cup \ldots$ for some sequence of natural numbers (n_i). This sequence can be coded by a real $f \in \omega^\omega$, which *enumerates* the basic open sets in U:

$$n \in \operatorname{ran}(f) \iff U_n \subset U.$$

Hence, to characterise all open subsets of X, it suffices to consider all reals $f \in \omega^\omega$ and the associated unions of basic open sets which they code.[3] This leads to the following characterisation of the class of open subsets of X:

$$\underset{\sim}{\mathbf{\Sigma}}^0_1(X) = \left\{ \bigcup \{U_{f(n)} \mid n < \omega\} \,\middle|\, f \in \omega^\omega \right\}.$$

By taking complements we obtain the class of closed sets

$$\underset{\sim}{\mathbf{\Pi}}^0_1(X) = \neg\, \underset{\sim}{\mathbf{\Sigma}}^0_1(X).$$

For $\alpha < \omega_1$, define

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X) = \left\{ \bigcup_{n<\omega} A_n \,\middle|\, A_n \in \underset{\sim}{\mathbf{\Pi}}^0_{\beta_n}(X) \text{ for some } \beta_n < \alpha \right\}$$

and we define again

$$\underset{\sim}{\mathbf{\Pi}}^0_\alpha(X) = \neg\, \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X).$$

Finally, sets that are both $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X)$ and $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$ are called $\underset{\sim}{\mathbf{\Delta}}^0_\alpha(X)$, hence

$$\underset{\sim}{\mathbf{\Delta}}^0_\alpha(X) = \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X) \cap \underset{\sim}{\mathbf{\Pi}}^0_\alpha(X).$$

We call these collections of sets **Borel pointclasses**: for each $\alpha < \omega_1$, the collection $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ (and $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$) is a Borel pointclass. Observe that the elements of each pointclass are *subsets* of the underlying Polish space, but that the

[3]This representation of open sets in terms of reals from ω^ω will be of interest in section 2.1.4, where we consider the *lightface hierarchy*.

pointclasses themselves are *not*. For example, the class of all open sets $\utilde{\mathbf{\Sigma}}^0_1(X)$ is a Borel pointclass, but no open subset of X is a Borel pointclass itself.

Definition 2.1.3. *For X Polish, a set $A \subset X$ is* **Borel** *if $A \in \utilde{\mathbf{\Sigma}}^0_\alpha(X)$ for some $\alpha < \omega_1$. The class of Borel subsets of X is denoted by $\mathcal{B}(X)$.*

It follows immediately that the class of Borel sets forms a σ-algebra: the class $\mathcal{B}(X)$ is closed under countable unions and complements. It is in fact the smallest σ-algebra containing the open sets, as it is generated by them.

The structure of the Borel sets lends itself to proofs by induction: proving global properties of the class of Borel sets often reduces to an argument by induction which carries *local* properties through the pointclasses, and leads to *global* conclusions. For instance, Lemma 2.1.2 can be easily extended through all Borel sets by induction on ω_1, which shows that the Borel sets form a hierarchy described by the following diagram:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \utilde{\mathbf{\Sigma}}^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\mathbf{\Sigma}}^0_2 & & \\
\utilde{\mathbf{\Delta}}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\mathbf{\Delta}}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \subseteq & \cdots \\
 & \utilde{\mathbf{\Pi}}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \utilde{\mathbf{\Pi}}^0_2 & &
\end{tikzcd}$$

Since Borel sets are defined by countable sequences of operations, ω_1 is the maximal possible length for the Borel hierarchy. The following theorem ensures that the hierarchy is not trivial.

Theorem 2.1.4 ([72, II.22.4])**.** *Suppose X is uncountable Polish. Then for every $\alpha < \omega_1$, we have $\utilde{\mathbf{\Sigma}}^0_\alpha(X) \subsetneq \utilde{\mathbf{\Sigma}}^0_{\alpha+1}(X)$.*

For completeness, we mention that the Polish subspaces of Polish spaces are easily classifiable [72, I.3.11]:

Theorem 2.1.5 (Aleksandrov's theorem). *A subspace $Y \subset X$ of a Polish space X is Polish itself if and only if $Y \in \mathbf{\Pi}^0_2(X)$.*

2.1.1 Structural Properties of the Borel Hierarchy

Beyond the obvious stratification provided by the Borel hierarchy, there are deeper structural properties to uncover, which shape not only the hierarchy as a whole but also lend order to the sets that appear in it. In this section we focus on *global* properties of the Borel hierarchy: those which give structure to the class of all Borel sets. All can be found in [72].

We begin with a number of properties showing that uncountable Polish spaces are not simple. Recall that a Polish space is **perfect** if all of its points are limit points. The following two results are classical.

Lemma 2.1.6 ([72, I.6.2]). *Every uncountable perfect Polish space contains a homeomorphic copy of 2^ω.*

Lemma 2.1.7 ([72, I.7.5]). *Every uncountable perfect Polish space contains a homeomorphic copy of ω^ω.*

The next theorem is known as the **Cantor-Bendixson theorem**. It plays an important role in determining the complexity of subsets Polish spaces, by introducing a rank function on its subsets, the **Cantor-Bendixson derivative** and its associated **Cantor-Bendixson rank** [72, 6.B, 6.C].

Theorem 2.1.8 (Cantor-Bendixson theorem). *Every Polish space can be written as the disjoint union of a countable open set and a perfect set.*

We note two consequences. Firstly, it follows from Aleksandrov's Theorem 2.1.5 that every uncountable Polish space—perfect or not—contains a copy of 2^ω and ω^ω: by the Cantor-Bendixson theorem, every Polish space contains a closed—and hence $\utilde{\mathbf{\Pi}}^0_2$—perfect subset, which is Polish itself by Aleksandrov's theorem. Now Lemmas 2.1.6 and 2.1.7 yield the result. Secondly, every uncountable Polish space has cardinality continuum: Lemma 2.1.6 yields one direction: take the uncountable perfect subset, which now must contain a copy of Cantor space. The next lemma implies the other direction.

Lemma 2.1.9 ([72, I.4.14])**.** *Let X be uncountable Polish*[4]*. Then X can be topologically embedded into the Hilbert cube* $[0,1]^\omega$.

Remark. This result gives rise to a notion first isolated by Gödel [68, p. 133] in his attempts to resolve CH: that of the **perfect set property** (PSP). A subset A of a Polish space X has the PSP if A is countable or contains a perfect subset. Importantly, if A has the PSP then A cannot be a counterexample to CH, as it contains a copy of 2^ω, and hence has cardinality $2^{\aleph_0}$. Therefore, Aleksandrov's Theorem 2.1.5 together with the Cantor-Bendixson Theorem 2.1.8 imply that no $\utilde{\mathbf{\Pi}}^0_2$ (and no $\utilde{\mathbf{\Sigma}}^0_2$) can violate CH. In fact more is true: every Borel set has the PSP, which is due to Aleksandrov and Hausdorff [72, II.13.6], proving that counterexamples to CH cannot be constructed definably. In fact, Souslin showed that every *analytic set*—which we are to introduce later—has the PSP. However, once AC is involved, one can construct sets that do not have the PSP, such as Bernstein sets (cf. section 2.1.3).

2.1.2 Beyond the Borel Hierarchy

Are there sets of reals that are not Borel? The answer to this question jumpstarted the development of descriptive set theory as we know it today,

[4] In fact, separable metrisable suffices.

and stems from Lebesgue's erroneous assumption that the Borel hierarchy is closed under continuous images—in particular, he thought to have proved that Borel sets are closed under projections [119, II.17]. After Souslin noticed this mistake, he constructed a non-Borel set [72, II.14.2]. The structure of these more complicated sets—those *beyond* the Borel hierarchy—begins with the **analytic sets**, whose pointclass we denote by $\underset{\sim}{\mathbf{\Sigma}}^1_1$.

Definition 2.1.10. *A subset $A \subset X$ of a Polish space X is* **analytic** *if A is the continuous image of some Polish space Y: there exists a continuous function $f\colon Y \to X$ such that $A = f[Y]$.*

The study of analytic sets can be simplified by passing to convenient equivalent definitions. Firstly, it suffices to consider continuous images of ω^ω to classify all analytic subsets; this follows since every non-empty Polish space is in fact itself a continuous image of ω^ω [72, I.7.9]. Secondly, there exists a versatile characterisation of analytic sets in uncountable Polish spaces which we wish to highlight (for a proof see e.g. [145, Prop 4.1.1]).

Theorem 2.1.11 ([72, II.14.3])**.** *A subset $A \subset X$ of a Polish space X is analytic if and only if any (all) of the following conditions are satisfied:*

1. *There is a Polish space Y and a Borel set $B \subset X \times Y$ such that $A = \mathrm{proj}_X(B)$.*
2. *There is a closed set $F \subset X \times \omega^\omega$ such that $A = \mathrm{proj}_X(F)$.*
3. *There is a $\underset{\sim}{\mathbf{\Pi}}^0_2$ set $G \subset X \times 2^\omega$ such that $A = \mathrm{proj}_X(B)$.*

Expressing analytic sets in terms of projections also lends meaning to the use of the symbol $\underset{\sim}{\mathbf{\Sigma}}^1_1$: projections quantify existentially over second-order objects (reals from ω^ω) instead of just over first-order objects (open sets, which can be enumerated by ω since Polish spaces are second-countable).

Item 2 is most useful, as it plays a significant role when $X = \omega^\omega$. To develop this, we need the notion of set-theoretical trees. A set $T \subset A^{<\omega}$ is a **tree** if for every $\sigma \in T$ and every initial segment $\tau \prec \sigma$ we have $\tau \in T$. If $A = B \times C$ is a product space itself, then trees on A are defined in the same way; here we also demand that if $(\sigma, \tau) \in T$ then σ and τ have the same length; and (σ', τ') is an initial segment of (σ, τ) if and only if σ' is an initial segment of σ, and the same holds for τ' and τ. A tree is **pruned** if it has no dead ends. The set of **paths through the tree** T is given by

$$[T] = \{x \in A^\omega \mid (\forall n)(x \restriction n \in T)\}.$$

Noting that $\omega^\omega \times \omega^\omega$ and $(\omega \times \omega)^\omega$ are topologically isomorphic shows that the closed sets of $\omega^\omega \times \omega^\omega$ are of the form $[T]$ for some pruned tree T on $\omega \times \omega$.Hence the case $X = \omega^\omega$ in item 2 now reduces to the following: the class of analytic subsets of ω^ω can be identified by the class of trees on $\omega \times \omega$. To formally relate the analytic sets to the Borel hierarchy, we note that:

Lemma 2.1.12 ([72, II.13.7])**.** *Every Borel set is analytic.*

However, by diagonalising against a universal set, Souslin proved:

Theorem 2.1.13 ([72, II.14.2])**.** *Let X be an uncountable Polish space. Then there exists a set $A \subset X$ which is analytic yet not Borel.*

The projective hierarchy does not end at the analytic sets. By considering complements, we arrive at the **co-analytic sets**, denoted by $\utilde{\mathbf{\Pi}}^1_1$. We also denote by $\utilde{\mathbf{\Delta}}^1_1$ the pointclass of sets which are both analytic and co-analytic. Due to a fundamental theorem of Souslin [142] (see also [72, II.14.11]), its place relative to the Borel hierarchy is easily described:

Theorem 2.1.14 (Souslin's Theorem)**.** *A set is Borel if and only if it is $\utilde{\mathbf{\Delta}}^1_1$.*

We pass to more complicated sets via projections: the $\mathbf{\Sigma}^1_2$ sets are obtained by taking projections of $\mathbf{\Pi}^1_1$ sets of $X \times \omega^\omega$ onto the first coordinate, and so on. This yields the **projective hierarchy**; its structure looks exactly the same as the finite-level Borel structure:

$$
\begin{array}{ccccccccc}
 & & \mathbf{\Sigma}^1_1 & & & & \mathbf{\Sigma}^1_2 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & & \\
\mathbf{\Delta}^1_1 & & & & \mathbf{\Delta}^1_2 & & & \subset & \cdots \\
 & \supsetneq & & \subsetneq & & \supsetneq & & & \\
 & & \mathbf{\Pi}^1_1 & & & & \mathbf{\Pi}^1_2 & &
\end{array}
$$

As in the Borel case, every inclusion is proper by virtue of the existence of universal sets [72, V.37.7]. We record a few closure properties: the classes $\mathbf{\Sigma}^1_n$ are closed under countable intersections and unions, existential quantification over Polish spaces, and continuous pre-images; the $\mathbf{\Pi}^1_n$ classes are also closed under countable intersections and unions and continuous pre-images, and under universal quantification over Polish spaces [72, V.37.1].

While the analytic sets are quite well-behaved, the co-analytic sets are more complicated. For instance, every analytic set:

- has the PSP [72, III.29.1]
- has the property of Baire [72, II.21.6]
- is Lebesgue measurable.[5]

On the other hand, one can construct assuming $V{=}L$ a so-called **thin** co-analytic set: an uncountable co-analytic set that contains no perfect subset, and hence does not have the PSP [119, Thm 23.1].[6]

[5] These results are all due to Lusin [87] and were published in the same issue as Souslin's important [142]; the latter result on Lebesgue measurability is attributed to Souslin, much to the chagrin of Aleksandrov; see [68, p. 148] for historical details.

[6] This is originally due to Gödel [49]; see also [140], where, interestingly, it also shown

2.1.3 Definable Counterexamples and Games

Many properties of sets of reals can be reduced to perfect information two-player games. Take the PSP for example [72, II.21.2]: for any $A \subset \omega^\omega$ there exists a game G_A such that if either player has a winning strategy in the game G_A—we say that G_A is *determined*—then A has the PSP. The same is true for Lebesgue measurability and the property of Baire. In general: the more games are determined, the more sets of reals behave nicely.

Axiomatically, determinacy for all Borel sets is provable: this is Martin's celebrated *Borel determinacy* theorem [103, 104]. However, we cannot prove more: determinacy for analytic sets is a large cardinal property [63, III.33.19]. Ignoring provability, we can go further: the **axiom of determinacy** AD postulates that *every* set of reals is determined [63, III.33]. Hence: under AD *all* sets of reals have the PSP, are Lebesgue measurable, and so on [63, III.33.3]. This structural order comes at a cost: AD is incompatible with AC, which can be seen by an easy diagonalisation argument [63, III.33.1].

This structural impact of AD (and its orthogonal behaviour to AC) follows a particular pattern: there exist properties P for sets of reals

- which are true for all Borel (or sometimes analytic) sets in ZFC;
- which are true for *all* sets if AD holds;
- which are *not* true for all sets if AC holds; and
- for which there exists a *definable counterexample* if $V{=}L$ holds.

The search for definable counterexamples (under $V{=}L$ and otherwise) is a fruitful line of contemporary research: definable counterexamples measure precisely the optimality of existence theorems. E.g. if there exists a $\undertilde{\mathbf{\Pi}}^1_1$-set

that the following are equivalent over ZFC: every $\undertilde{\mathbf{\Pi}}^1_1$ set of reals has the PSP if and only if every $\undertilde{\mathbf{\Sigma}}^1_2$ set of reals has the PSP.

not having property P, yet P provably (under some prudent choice of axioms) holds for all $\underset{\sim}{\mathbf{\Sigma}}^1_1$-sets, then there cannot be a proof of P for all Borel sets.

The PSP is a classical example. Souslin proved that every analytic set in an uncountable Polish space has the PSP [72, II.14.13]. However, invoking a weak form of the axiom of choice, in 1908 Felix Bernstein showed the existence of sets failing the PSP (cf. [68, 11.4], [72, I.8.24], [127, Thm 5.3]). And using $V{=}L$, Gödel [49] constructed an uncountable co-analytic set that does not contain a perfect subset (see also [68, 13.12] for a proof), hence showing that the PSP is optimal for analytic sets. On the other hand, we already mentioned that AD implies that *every* set has the PSP; this is due to Mycielski and Swierczkowski [124].[7] So to recap: the PSP holds for all analytic sets; holds for all sets under AD; fails for some set under AC; and fails for a co-analytic (and hence optimal) set under $V{=}L$.

This sensitivity to set-theoretical axioms, and the search for definable counterexamples to determine sharpness, is the motivation behind our work on Marstrand's theorem in part II. While Marstrand's theorem holds for all analytic sets, we show under $V{=}L$ how to construct a definable counterexample: a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set for which Marstrand's theorem fails.

There is also a philosophical argument to be made. Since theorems of this type assert the truth for all sets under AD, one can argue that, depending on the set-theoretical context and one's willingness to assume stronger structural axioms (such as AD), in fact *all* sets satisfy the asserted property. In particular, it is arguable that there is nothing special about Borel sets, as

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[7]More historical details can be found in [68, p. 377]. Also, for an overview of general questions on perfect set properties we recommend [68]; for recent advances and generalisations see for instance [114].

structural properties inherited by their construction can be forced upon all sets of reals provided one is willing to change the set-theoretical landscape. This case is made, for instance, by Antonio Montalbán [121, p. 1214] who writes in a footnote that "[w]e state these [aforementioned] results [about the Wadge degrees] in terms of Borel sets because that is how much we can prove in ZFC, but they are not really about Borel sets. All of this holds for all constructible sets in $L(\mathbb{R})$[8] if one assumes the large-cardinal hypothesis and for all sets if one assumes [AD] and forgets about [AC]."

These phenomena highlight the importance of set-theoretical axioms in classical mathematics—an opinion shared by the author of this thesis.

2.1.4 Effectivising the Theory

Recall that in the definition of $\underset{\sim}{\mathbf{\Sigma}}^0_1(\omega^\omega)$, we consider *all* families of basic open sets: if $\{U_n \mid n < \omega\}$ is a basis of ω^ω, the union of any sequence of basic open sets is open. What if we only allow sequences of basic open sets which can be coded by a *computable function*? In other words, what happens if we only consider those open sets $U \subset X$ which admit a computable $f\colon \omega \to \omega$ for which $U = \bigcup\{U_{f(n)} \mid n < \omega\}$? Clearly not every open set is caught by such an enumeration. The resultant **effectivised** theory is of interest to set theorists and computability theorists alike [123, 48].

The effectivisation yields the **lightface Borel pointclasses**, which combine into the **effective Borel hierarchy**. At its lowest level appear the **effectively open**, or (lightface) Σ^0_1 sets. By carrying the effective construction of Borel sets further (i.e. only allowing *computable* unions), we can

[8] $L(\mathbb{R})$ is the smallest inner model of ZF containing the reals; see [63, p. 193]

describe all lightface classes Σ^0_α, Π^0_α, and Δ^0_α over ω^ω. Formally, constructing these classes requires care at the limit steps; see [48, Dfn 1.5.3] for details. Importantly, every $A \in \Sigma^0_\alpha$ has a **hyperarithmetic index**: a code describing the effective operations of union and complementation in the transfinite construction of A. Note that a hyperarithmetical index cannot code a Borel set of Borel rank greater than ω_1^{CK}; hence the hierarchy stabilises there.

Definition 2.1.15. *A set $A \subset \omega^\omega$ is* **hyperarithmetical** *if it has a hyperarithmetical index. The class of hyperarithmetical sets is denoted by* HYP.

Considering oracles yields the **relativised pointclasses** $\Sigma^{0,z}_\alpha$, $\Pi^{0,z}_\alpha$, and $\Delta^{0,z}_\alpha$. These stabilise at $\omega_1^{\mathsf{CK}(z)}$, the least ordinal not computable from z. Since for any Borel set A there exists an oracle and a hyperarithmetical index that effectively knows the sequence of complementations and unions in A's construction, we can identify the classical Borel hierarchy effectively.

Theorem 2.1.16 ([48, Thm 1.5.4.]). *$A \subset \omega^\omega$ is Borel if and only if $A \in \Sigma^{0,z}_\alpha(\omega^\omega)$ for some $z \in 2^\omega$ and some $\alpha < \omega_1^{\mathsf{CK}(z)}$. In particular,*

$$A \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\omega^\omega) \iff A \in \Sigma^{0,z}_\alpha(\omega^\omega) \textit{ for some } z \in 2^\omega \textit{ with } \omega_1^{\mathsf{CK}(z)} > \alpha.$$

Effectivisations can be carried into the projective hierarchy:

Definition 2.1.17. *A set $A \subset \omega^\omega$ is Σ^1_1 if and only if there exists a computable tree T on $\omega \times \omega$ such that $A = \mathrm{proj}_1(T)$.*

The following theorem is due to Kleene [73]; relativising yields a proof of Souslin's Theorem 2.1.14.

Theorem 2.1.18 (Kleene). $\mathsf{HYP} = \Delta^1_1$

While HYP describes *sets* of reals, we can equally consider individual reals as hyperarithmetical objects. These are motivated by carrying the Turing jump beyond the finite ordinals: we build the sequence

$$\emptyset, \emptyset', \emptyset'', \dots, \emptyset^{(\omega)}, \emptyset^{(\omega+1)}, \dots$$

which runs up[9] to ω_1^{CK}. The construction of this hierarchy, which also requires care at the limit steps, is carefully carried out in [2, 5.1] As a result, we may speak of $\emptyset^{(\alpha)}$ for any ordinal $\alpha < \omega_1^{\mathsf{CK}}$.

Definition 2.1.19. *$A \subset \omega$ is* **hyperarithmetical** *if $A \leq_T \emptyset^{(\alpha)}$ for some $\alpha < \omega_1^{\mathsf{CK}}$. The class of hyperarithmetical subsets of ω is denoted by* $\mathsf{HYP}(\omega)$.

Using transfinite jumps, the following is useful to show that a set is Borel.

Theorem 2.1.20 (cf. [24, Prop 2.3])**.** *$A \subset \omega^\omega$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\alpha}$ iff there exists $z \in 2^\omega$ with $\omega_1^{\mathsf{CK}(z)} > \alpha$ and a computable procedure Φ such that for every $x \in \omega^\omega$:*

$$x \in A \iff \Phi\Big(\big(x \oplus z\big)^{(\alpha)}\Big) \text{ halts}$$

Definition 2.1.21. *Let $A, B \subset \omega$. Then A* **is hyperarithmetical in** *B if*

$$\Big(\exists \alpha < \omega_1^{\mathsf{CK}(B)}\Big)\Big(A \leq_T B^{(\alpha)}\Big).$$

We write $A \leq_h B$. The relation $\leq_h$ is easily seen to be reflexive, transitive, and coarser than $\leq_T$ ($\emptyset \equiv_h \emptyset'$ yet $\emptyset <_T \emptyset'$). Theorem 2.1.18 applies[10] to ω:

Theorem 2.1.22 (Kleene [73])**.** $\mathsf{HYP}(\omega) = \Delta^1_1(\omega)$

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[9] Any ordinal below ω_1^{CK} is computable, or equivalently, has an ordinal notation [101, 74]. The choice of ordinal notation is irrelevant [143].

[10] Here, Σ^1_1 sets are defined by formulas with leading (and no other) second-order quantifiers, followed by arithmetical relations; cf. [2, 5.2] for details.

We now carry the construction of effective descriptive set theory on ω^ω a little further. Firstly, observe that reals in Baire space ω^ω can easily be identified with subsets of ω: for $f \in \omega^\omega$ define

$$\hat{f} = \{(x, f(x)) \,|\, x < \omega\}$$

as a subset of ω, via coding of pairs. Hence we call $f \in \omega^\omega$ **hyperarithmetical** if $\hat{f} \in \mathsf{HYP}(\omega)$. Further, by relativising $\mathsf{HYP}(\omega) = \Delta^1_1(\omega)$ we also see that $f \leq_h g \iff \hat{f} \in \mathsf{HYP}^{\hat{g}}(\omega) \iff f \in \Delta^{1,\hat{g}}_1(\omega)$.

Properties of interest in hyperarithmetic theory include:

- *representations*: e.g. sets have useful normal forms in terms of combinatorial objects such as trees [119, Thm 17.4])
- *reductions*: e.g. pairs of Π^1_1 sets can be simplified by being reduced to pairs of disjoint Π^1_1 sets [119, Section 28]); and
- *uniformisation properties*: e.g. every Π^1_1 relation R can be *uniformised*, there exists a Π^1_1 set containing exactly one witness for each input; (see [119, Thm 22.1] and Kondô's original proof [77]).

These lightface results recover virtually all classical results. E.g. let $X \in \{\omega, \omega^\omega\}$.[11] The reduction property of Π^1_1 sets implies Σ^1_1 *separation*: if A and B are disjoint Σ^1_1 sets then there exists a Δ^1_1 set separating them. By relativising the theorem of Π^1_1 reductions [48, Thm 1.7.1] we obtain Lusin's separation theorem, which asserts the same separation property for $\utilde{\boldsymbol{\Sigma}}^1_1$ sets; on ω^ω, this proves Souslin's Theorem 2.1.14.

The following example [119, Section 29] will be useful in future sections. Suppose $X \in \{\omega, \omega^\omega\}$, and that the following exist:

- a Π^1_1-set of *codes* $C \subset \omega \times \omega^\omega$; and

[11] In fact, the results hold for any space of the form $\omega^m \times (\omega^\omega)^n$ for $m, n < \omega$.

- two sets of *descriptions* $S, P \subset (\omega \times \omega^\omega) \times X$ where $S \in \Sigma^1_1$ and $P \in \Pi^1_1$

such that for every code $(e, u) \in C$

$$P_{e,u} = \{x \in X \mid (e, u, x) \in P\} = \{x \in X \mid (e, u, x) \in S\} = S_{e,u},$$

and if $u \in \omega^\omega$ and $D \subset X$ is $\Delta^{1,u}_1$ then there exists $e < \omega$ such that

$$P_{e,u} = D = S_{e,u}.$$

We then say that e is a **$\Delta^{1,u}_1$-code** for D. If $\mathcal{D}(u)$ denotes the class of $\Delta^{1,u}_1$-codes, then note that

$$e \in \mathcal{D}(u) \iff P_{e,u} = S_{e,u} \iff (e, u) \in C \tag{2.1}$$

which is a Π^1_1-relation by definition of C. Such descriptions provably exist [119, 29.1]; this is (a version of) the **Spector-Gandy Theorem** [144, 47].

Theorem 2.1.23 (Spector-Gandy Theorem)**.** *Δ^1_1-codes exist.*

Corollary 2.1.24 ([119])**.** *If $A \subset (\omega^\omega)^2$ is Π^1_1, so is the set of x for which*

$$(\exists y \leq_h x)((x, y) \in A).$$

Proof. Suppose $x \in \omega^\omega$ and consider $\hat{x}$. Now work in ω; we take care of the reverse coding by only picking witnesses $y \in 2^\omega$ that are graphs of reals in ω^ω. Since $\mathsf{HYP}(\omega) = \Delta^1_1(\omega)$, we know "$\exists y \leq_h x$" is equivalent to "$\exists y \in \Delta^{1,\hat{x}}_1(\omega)$". So $(\exists y \leq_h x)((x, y) \in A)$ if and only if there exists $e < \omega$ for which:

- e is a $\Delta^{1,\hat{x}}_1$-code for some set $D \subset \omega$ (a Π^1_1-predicate by eq. (2.1)) and D is the graph of a total function from ω to ω (this latter condition is arithmetical, thus does not contribute to the complexity); and
- for all $y \subset \omega$, if $D = y$ then $(x, y) \in A$.

Verifying whether $D = y$ is Π^1_1 since e is a $\Delta^{1,\hat{x}}_1$-code: D has descriptions in terms of $S \in \Sigma^1_1$ and $P \in \Pi^1_1$, and so

$$y = D \iff (\forall n < \omega)(n \in y \to (e, x, n) \in P \wedge (e, x, n) \in S \to n \in y). \quad \square$$

We close by connecting the constructible universe L to hyperarithmetical theory (cf. [4, IV.3] and [135, III.9.12] for details; [18, Thm 3.6.8] for a proof).

Theorem 2.1.25. $\mathsf{HYP}(\omega) = L_{\omega_1^{\mathsf{CK}}} \cap 2^\omega$.

This identification allows us to use set-theoretical tools in the classification of hyperarithmetical reals. The **self-constructibles**, investigated by Kechris [71], Guaspari, and Sacks [134] independently, are an example:

$$\mathcal{C}_1 = \left\{ \alpha \in \omega^\omega \,\middle|\, \alpha \in L_{\omega_1^{(\alpha)}} \right\}.$$

Lemma 2.1.26. $y \leq_h x \iff y \in L_{\omega_1^{(x)}}[x]$. *If* $x \in \mathcal{C}_1$ *then* $L_{\omega_1^{(x)}}[x] = L_{\omega_1^{(x)}}$.

Proof. The first part is the natural relativisation of the fact that x is hyperarithmetical (i.e. $x \leq_h \emptyset$) if and only if $x \in L_{\omega_1^{\mathsf{CK}}}$ [135, A.II.7.3, A.III.9.12]. The latter follows from the definition of $\mathcal{C}_1$. Clearly, $L_{\omega_1^{(x)}}[x] \supset L_{\omega_1^{(x)}}$; conversely, if $x \in \mathcal{C}_1$ then $x \in L_\alpha$ for some $\alpha < \omega_1^{(x)}$. By the first part and the definition of the construction of the hierarchy $L[x]$, we obtain: if $y \in L_{\omega_1^{(x)}}[x]$, then $y \in L_\beta[x]$ for some $\beta < \omega_1^{(x)}$. Let $\gamma > \alpha$ be the minimal limit, and assume w.l.o.g. that $\gamma \geq \beta$. Then, by carrying out the same construction as in $L_{\omega_1^{(x)}}[x]$, we see that $y \in L_{\gamma+\beta} \subset L_{\omega_1^{(x)}}$, as needed. $\square$

The class $\mathcal{C}_1$ is the largest **thin** (or **scattered**) Π^1_1 set (it contains no perfect subset) [71, 134], and it **traces the computable reals** [71]:

$$L \cap \omega^\omega = \{\alpha \mid (\exists \beta)(\beta \in \mathcal{C}_1 \wedge \alpha \leq_T \beta)\}.$$

If $V{=}L$ and X is $\boldsymbol{\Pi}^1_1$ then X is cofinal in HYP iff $X \cap \mathcal{C}_1$ is [17, Lem 3.2].

2.2 The Constructible Universe

In part I we use *higher computability theory*, a natural extension of computability theory to uncountable domains. Instead of working on ω we focus on initial segments of the constructible universe L. As higher computability theory uses the L-hierarchy, we include a brief overview of the relevant theory. For more details, we recommend [80, Chapters 5, 6].

The axioms of ZFC include $\mathsf{Power\ Set}$: for every x there exists y which is exactly the set of subsets of x, denoted $\mathcal{P}(x)$. By transfinite recursion, we can then construct the **cumulative hierarchy** V: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit. Using the axiom $\mathsf{Foundation}$, one can now show that every set x belongs to V. Gödel's insight [50] was the following: by allowing the definable subsets into the next level of the hierarchy, his class L has more structure—so much so that L decides many combinatorial set-theoretical questions. Most famously, AC and GCH are true in L [49, 50].

However, there is a sleuth of other combinatorial properties which are useful in the study of models of $V{=}L$—models in which every set is constructible. These tools include the ubiquitous *condensation lemma* as well as Jensen's fine structure theory [64, 63] which we make use of in chapter 4.

2.2.1 Definable Sets and Condensation

To characterise definable subsets of a structure, one can argue via formulas. Since formulas are not sets, their coding, and that of the satisfaction relation, is required, which yields a perfectly fine—yet very intricate—path towards defining definable sets. A faster approach is given by using **Gödel functions**. These capture logical notions such as quantifiers and negations in terms of functions on sets; the definable subsets of a transitive set x are then given by

$$\mathrm{def}(x) = \mathrm{cl}(x \cup \{x\}) \cap \mathcal{P}(x)$$

where $\mathrm{cl}(x)$ denotes the closure of the set x under Gödel operations. This gives rise to the constructible hierarchy: put $L_0 = \emptyset$, $L_{\alpha+1} = \mathrm{def}(L_\alpha)$, and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ if λ is a limit. The following are now easily shown:

- every L_α is transitive [80, VI.1.6];
- $L_\alpha \cap \mathrm{ON} = \alpha$ [80, VI.1.9];
- $|L_\alpha| = |\alpha|$ if AC holds [80, VI.1.14]; and
- L models ZFC: every axiom of ZFC relativised to L is true [80, VI.2.1].

We single out one very useful[12] property of constructibility theory: Gödel's **condensation lemma** (for a proof see [25, II.5.2]), which for instance is used to show GCH in L [63, II.13.20]. We first recall the **Mostowski collapse**. Say a structure $(x, \in)$ is **extensional** if whenever $v, w \in x$ and $v \neq w$ then $v \cap x \neq w \cap x$. The following is classical (see [63, I.6.15] for a proof):

Lemma 2.2.1 (Mostowski collapsing lemma). *Every extensional structure $(x, \in)$ is isomorphic to a unique transitive structure $(y, \in)$ via the unique isomorphism π mapping $v \mapsto \{\pi(w) \mid w \in v \cap x\}$.*

Theorem 2.2.2 (Condensation lemma). *Let $(M, \in)$ be an elementary[13] substructure of $(L_\alpha, \in)$. Then $M \cong L_\beta$ for some $\beta \leq \alpha$, witnessed by the Mostowski collapse π.*

More is true if M is transitive: since the Mostowski collapse fixes transitive sets point-wise, elementary equivalence can be replaced by equality (this

[12]Condensation is not without reason called "arguably the most important single result in constructibility theory (as far as applications are concerned)" [25, p. 80]

[13]In fact it suffices for M to be a Σ_1 elementary substructure for the condensation lemma to hold [63, Section 13.17].

holds for all $\alpha \geq \omega$ [63, p. 188]; the case $\alpha = \omega_1$ is proven in [25, II.5.10]):

Corollary 2.2.3. *Let $(M, \in)$ be an elementary substructure of $(L_\alpha, \in)$, and suppose M is transitive. Then $M = L_\beta$ for some $\beta \leq \alpha$.*

We will use both condensation and the Mostowski collapse repeatedly in chapter 4. Now, we close this section with a simple result concerning closure properties of well-behaved sequences of sets (which holds outside of L as well). Recall that a subset E of a cardinal κ is **stationary** if whenever $C \subset \kappa$ is closed unbounded then $E \cap C$ is non-empty.[14]

Definition 2.2.4. *Let κ be regular. A sequence $\langle C_\alpha \rangle_{\alpha<\kappa}$ is called an* **ice sequence on** *κ if:*

- *$C_\alpha \subset \kappa$ and $|C_\alpha| \leq |\alpha|$ for all $\alpha < \kappa$;*
- *if $\alpha < \beta$ then $C_\alpha \subset C_\beta$; and*
- *if λ is a limit then $\bigcup\{C_\alpha \mid \alpha < \lambda\} = C_\lambda$.*

The following is a well-known closure property which we use repeatedly in chapter 4, when we consider approximations to free abelian groups.

Lemma 2.2.5. *Suppose κ is regular, $\langle C_\alpha \rangle_{\alpha<\kappa}$ and $\langle D_\alpha \rangle_{\alpha<\kappa}$ are ice, and that $C_\kappa = D_\kappa$. Then the set $\{\alpha < \kappa \mid C_\alpha = D_\alpha\}$ is club in κ.*

Proof. Closedness follows immediately from the fact that both sequences are ice, and hence increasing and in particular continuous. For unboundedness, fix $\alpha_0 < \kappa$. Proceed by recursion: given α_n, find the least $\beta > \alpha_n$ such that

$$C_{\alpha_n} \cup D_{\alpha_n} \subset C_\beta \cap D_\beta.$$

[14]Measure-theoretically, one can think of closed unbounded sets as sets of full measure, while stationary sets have positive measure, since the clubs of a regular uncountable κ form a κ-complete filter.

Such a β always exists: since $C_\kappa = D_\kappa$ and $|C_{\alpha_n}| < \kappa$, there exists $\beta' < \kappa$ such that $C_{\alpha_n} \subset D_{\beta'}$; and the same is true with the roles of C_κ and D_κ reversed for some $\beta'' < \kappa$. Thus choosing $\alpha_{n+1} = \max\{\beta', \beta''\}$ does the trick. Now define $\alpha = \sup \alpha_n$. It is now easily seen that $C_\alpha = D_\alpha$, for

$$x \in C_\alpha \iff (\exists n < \omega)(x \in C_n) \iff (\exists n < \omega)(x \in D_{n+1}) \iff x \in D_\alpha$$

which is as required. Further, since κ is regular, we know that $\alpha < \kappa$. $\square$

2.2.2 Jensen's Fine Structure

Ronald Jensen pioneered the research of what is now called the **fine structure of** L [64]. Fundamental to its investigation are combinatorial properties, such as the class of **rudimentary functions** J, which we use in chapter 4. We outline its motivation below. Further sources beyond Jensen's original treatment recommended to the interested reader include [63] and [18].

The statement $V{=}L$ gives rise to a subtlety. Recall that $V{=}L$ is short for the statement "$(\forall x)(\exists \alpha)(x \in L_\alpha)$". This does not imply "$(\forall \alpha)(V_\alpha = L_\alpha)$", though, so if $V{=}L$ and $x \in V_\alpha \setminus L_\alpha$, what is the least β such that $x \in L_\beta$? What is the *index* of x? As L is constructed in a bottom-up process, understanding the structure of L often reduces to examining the levels L_α. For instance, in the proof of GCH in L one isolates a bound for the index of each subset, using condensation. However, solving the index-problem in L "uniformly" is difficult: while indices can often be bounded (cf. condensation) the weak closure properties of the L_α's prohibit an obvious *finer* differentiation. E.g. the L_α's are not closed under unordered pairs, which prohibits simple stage-by-stage arguments to compute the index explicitly.

This changed with Ronald Jensen's work [64] to build a hierarchy that resembles the constructible hierarchy often enough (in terms of levels between the new hierarchy and the L-hierarchy agreeing), yet is closed under more desirable properties. This hierarchy of sets closed under Jensen's **rudimentary functions** is also called the **J-hierarchy**. By passing from L to J we obtain *finer* tools to examine the indices of sets, and hence the structure of L itself—**Jensen's fine structure** of L was born.

Analogous to the construction of L via Gödel operations, the J-hierarchy is generated by the rudimentary functions [64, 63]:

1. The basic rudimentary functions are given by
$$F(x_1, \ldots, x_n) = x_i, \quad F(x_1, \ldots, x_n) = \{x_i, x_j\}, \quad F(x_1, \ldots, x_n) = x_i \setminus x_j.$$
2. If G is a rudimentary function then so is
$$F(y, x_1, \ldots, x_{n-1}) = \bigcup\{G(z, x_1, \ldots, x_{n-1}) \mid z \in y\}.$$
3. Every composition of rudimentary functions is rudimentary.

We denote the **rudimentary closure** of $M \cup \{M\}$, where M is a transitive set, by $\mathrm{rud}(M)$. We collect some facts about the J-hierarchy [63, Section 27]: every J-level is transitive, and the J-hierarchy is cumulative. One can also verify that $J_\alpha \subset V_{\omega\cdot\alpha}$, and even more finely, that $J_\alpha \cap \mathrm{ON} = \omega \cdot \alpha$. Further, we have a connection with L:

Lemma 2.2.6. *If M is transitive then* $\mathrm{rud}(M) \cap \mathcal{P}(M) = \mathrm{def}(M)$.

Hence we are actually just redefining L, possibly stretching the elements of L out along the J-hierarchy, which is defined naturally: $J_0 = \emptyset$, $J_{\alpha+1} = \mathrm{rud}(J_\alpha)$, and if λ is a limit ordinal then $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$. This hierarchy is also called the **Jensen hierarchy**, in Ronald Jensen's honour.

Many arguments of the fine structure theory reduce to combinatorial principles of the J-hierarchy itself. This analysis includes the investigation of ordinals and their "strength" in the following sense: how much does J_α know to be true about a given set x? For instance, x could be a cardinal according to some J_α, but some $\beta > \alpha$ might know better.

The class E, due to Jensen [64, §5], is an example of this phenomenon. It will play a major role in our arguments in chapter 4.

Theorem 2.2.7 ([64, Thm 5.1] ($V{=}L$)). *There exists a class E of limit ordinals and a sequence $\{C_\lambda \mid \lambda \in \mathrm{ON}$ is singular limit$\}$ such that:*

- *C_λ is club in λ;*
- *if $\gamma < \lambda$ is a limit of C_λ then (a) γ is singular, (b) $\gamma \notin E$, and (c) $C_\gamma = C_\lambda \cap \gamma$;*
- *if κ is regular and uncountable then $E \cap \kappa$ is stationary in κ.*

It follows directly from the properties of E that if κ is regular then $E \cap \kappa$ is stationary in κ. If κ is singular, however, then C_κ is club in κ, yet no limit of (ordinals in) C_κ is in E. Hence we have established that:

Corollary 2.2.8. *Let κ be cardinal. Then $E \cap \kappa$ is stationary in κ if and only if κ is regular.*

As for the construction of E, it turns out that E can be easily axiomatised.

Definition 2.2.9. *The class E from Theorem 2.2.7 is exactly the class of those limit ordinals satisfying the following: $\alpha \in E$ if and only if:*

- *$J_\beta \vDash \mathsf{ZF}^-$;*
- *α is the largest cardinal in J_β, and regular in J_β; and*
- *there exists a parameter $p \in J_\beta$ such that J_β is the smallest elementary substructure X of J_β such that $p \in X$ and $\alpha \cap X$ is transitive.*

This class will play an important role in chapter 4, where it is expressed in terms of the L-levels. The verification of the properties of E requires a complicated proof, which can be found in [64].

2.3 Transfinite Recursions for Sets of Reals

We give an example of a classical recursive construction, which produces a set of reals with prescribed properties, using AC. We then introduce Zoltán Vidnyánszky's theorem on producing co-analytic sets under $V{=}L$ [152], which allows us to control the projective complexity of certain recursively constructed sets of reals. This result will be a fundamental tool in both of our Theorems 6.4.1 and 6.5.1 in part II.

Transfinite recursive constructions belong to the basic toolkit of any logician, especially to that of computability (or *recursion*) theorists. These constructions are often of a very specific form: a desired global property (for example a certain property of a set of reals, or a Turing degree) is reduced to local *conditions* which can be satisfied one-by-one in a step-by-step process. It is then usually established that (1) every condition can be satisfied (which ensures that the conditions are not inherently contradictory), and (2) once all conditions are satisfied, then the desired global property holds.

While ubiquitous in classical computability theory—building incomparable degrees, jump inversion theorems, etc. are all instances of this process—we focus on set-theoretical applications of this technique: we construct[15] "strange subsets of $\mathbb{R}^n$". A typical example [20, Thm 6.1.2] is the *two-point set*: a set $A \subset \mathbb{R}^2$ which intersects every straight line in exactly two points.

[15] Many examples can be found in Ciesilski's [20, Part III], which we strongly recommend.

Its existence was first established by Stefan Mazurkiewicz [112][16], and we give the proof of [20], emphasising the typical components of recursive constructions of sets of reals.

Theorem 2.3.1 (Mazurkiewicz (AC)). *There exists a two-point set.*

Proof. We carry out a transfinite recursion along all straight lines in $\mathbb{R}^2$, whose collection we assume to be well-ordered: let $(L_\alpha)_{\alpha<\mathfrak{c}}$ be such a sequence. Each line is a condition, and we satisfy conditions in stages: L_α is *satisfied by* $X \subset \mathbb{R}^2$ if $|L_\alpha \cap X| = 2$. At stage $\alpha < \mathfrak{c}$, consider the pair (P_α, L_α), where L_α is the next condition and P_α is the set of all points we have already enumerated into our set. We assume that P_α is a *partial solution*; here, this means that there exists no $\gamma < \mathfrak{c}$ for which $|P_\alpha \cap L_\gamma| > 2$. Find $X_\alpha \subset \mathbb{R}^2$ so that

- $P_\alpha \cup X_\alpha$ satisfies the line L_α; and
- $P_\alpha \cup X_\alpha$ is still a partial solution.

Since, by assumption, no three points in P_α are collinear, $|L_\alpha \cap P_\alpha| \leq 2$. If it equals 2, then we are done. If not, consider the set $\mathcal{L}$ of all lines spanned by any non-degenerate pair of points in P_α. Since $|P_\alpha| \leq 2|\alpha| = |\alpha| < \mathfrak{c}$, we have $|\mathcal{L}| \leq |P_\alpha|^2 = |\alpha| < \mathfrak{c}$ (we overcount by considering *every* pair of points). Each $L \in \mathcal{L}$ intersects L_α at most once (if there existed L which met L_α twice, then $L = L_\alpha$ since all lines are assumed to be straight, so L_α is already satisfied). How many points are available to satisfy L_α without violating the partial solution P_α? This is a cardinality argument: $|L_\alpha \cap \bigcup \mathcal{L}| = |\bigcup_{L\in\mathcal{L}}(L_\alpha \cap L)| \leq |\alpha| < \mathfrak{c}$. Thus pick one (or two) points from $L_\alpha \setminus \bigcup \mathcal{L}$. $\square$

While this existence argument is straightforward, considering the com-

[16]See [113] for a French translation.

plexity of any two-point set constructed in this way is far from trivial. For instance, it is an open question first posed by Erdős [109] whether any two-point set can be Borel. This is not a coincidence: there are two hurdles typical for transfinite recursions which make recursive constructions producing Borel sets difficult to carry out. Firstly, transfinite recursions often have length $\mathfrak{c}$ and are hence *probably* not Borel, since every Borel set has a *countably* encoded Borel code [48, Dfn 1.4.1]. This was recently emphasised by Chad, Knight, and Suabedissen [13]. Secondly, AC usually features in recursive constructions: not only is the set of conditions necessarily well-ordered—which might already require some choice (but can be avoided sometimes; see e.g. [13])—but also, we satisfy conditions by *picking* suitable candidates.

In the particular case of two-point sets, progress on the necessity of AC have been made by Chad et al. [13], but it has been piecemeal. For instance, they showed that by picking a nicer well-ordering of conditions (i.e. lines in their context) one can circumvent the need for full choice and construct a two-point set in $\mathsf{ZF} + T$ where the axiom T asserts the existence of a certain well-orderable cardinal providing a nice representation of $\mathbb{R}$ [13, Thm 4.2]—the authors also claim that T is provable from $\mathsf{ZFC} + \mathsf{CH}$, showing consistency, but it is still open whether T can be proven from $\mathsf{ZF} + \neg\,\mathsf{AC}(\mathbb{R})$ alone.[17]

Progress has been made on the aforementioned complexity question, too: we know that there cannot be a $\utilde{\mathbf{\Sigma}}^0_2$ two-point set [81], and Mauldin showed that any analytic two-point set is Borel [118, Section 7]: if $(x, y) \notin S$ then the vertical line through $(x, 0)$ meets S in two points different from (x, y):

$$(x, y) \notin S \iff (\exists u, v)((x, u) \in S \wedge (x, v) \in S \wedge y \neq u \wedge y \neq v \wedge u \neq v).$$

[17] Arnold Miller showed in an unpublished note how to construct a two-point set in a model of ZF in which $\mathbb{R}$ is not well-orderable [120].

Now, if S is $\undertilde{\mathbf{\Sigma}}^1_1$ then so is $\neg S$. However, even this simplification has not resolved the long-standing question of possible complexities: the Borel-answer Erdős asked for remains elusive.

The interaction between AC and Borel-instances in the two-point set question above is typical. It is just one example of recursive constructions of sets of reals for which it is difficult to find a Borel solution—or to show that no Borel solution exists. Another is given by the partition of $\mathbb{R}^3$ into unit circles, for which many constructions are known using AC [66], as well as constructive ones provided one allows circles of varying radii [149]. However, both questions—whether AC is required, and whether there exists a Borel decomposition—remain open to the author's knowledge, only being aware of isolated remarks on the topic [57].

In essence, while the construction of sets in Polish spaces involving step-by-step processes of satisfying conditions is considerably simplified by transfinite recursion, ensuring that the complexity of the resultant sets remains low (in a descriptive set-theoretical sense) is rather difficult—and this difficulty can often be traced back to AC.

2.3.1 How to Ensure Co-analycity

So, what can we do to *control* the complexity of sets constructed by transfinite recursion? Recently, Zoltán Vidnyánszky [152] proved how to carry out transfinite recursions which produce co-analytic sets. He showed that if one can code the recursion into a co-analytic set (in a sense to be made precise later), then one only needs to focus on the candidates satisfying any given condition. If the sets of candidates are sufficiently complicated in the hyperarithmetical theory, the recursion can be carried out as normal, but the resultant set will be co-analytic. Evidently, this result is very powerful: it allows us to carry out recursions virtually identical to classical arguments, yet by augmenting ideas from computability theory we obtain more control, and hence stronger theorems. However, this expressiveness comes at a cost: so far, Vidnyánszky's theorem is only known to hold under $V{=}L$.

The theorem is a generalisation of ideas applied by A. Miller [118]—who used them to show the existence of co-analytic two-point sets, assuming $V{=}L$ [118, Thm 7.21]—but its very first use can be traced back to Erdős, Kunen, and Mauldin [32, Thms 13, 15], who showed, also assuming $V{=}L$, how to construct co-analytic sets of reals with peculiar measure properties.

As we will frequently handle partial solutions, we introduce some notation: if $X = \{x_\alpha \mid \alpha < \omega_1\}$ then we define

$$X \restriction \alpha = \{x_\beta \,|\, \beta < \alpha\}.$$

We are now ready to define what it means for a set to *code* a recursion.[18]

Definition 2.3.2. *Given $F \subset \mathbb{D}^{\leq\omega} \times [0, \pi/2] \times \mathbb{D}$, a set $X = \{x_\alpha \,|\, \alpha < \omega_1\}$ is* **compatible with** *F if the following exist:*

- *an enumeration $\{p_\alpha \,|\, \alpha < \omega_1\}$ of B; and*
- *an enumeration $\{A_\alpha \,|\, \alpha < \omega_1\} \subset \mathbb{D}^{\leq\omega}$ s.t. if $\alpha < \omega_1$ then $A_\alpha = X \restriction \alpha$*

such that for each $\alpha < \omega_1$ we have $(A_\alpha, p_\alpha, x_\alpha) \in F$.

Definition 2.3.2 formalises the idea of solving the recursion coded by F: each (A_α, p_α) codes a partial solution and a condition; x_α is a candidate satisfying p_α without violating A_α. Note each $A_\alpha \in \mathbb{D}^{\leq\omega}$ has order type $\leq \omega$, so all partial solutions are countable: if for every $\alpha < \omega_1$ we enumerate only

[18]This definition appears in [152, Dfn 1.2] in a slightly more general form; for our purposes we will not need more than presented here.

countably many points, then $X \restriction \alpha$ will be countable. This often simplifies arguments (cf. Theorem 6.4.1). As we assume $V{=}L$ below, CH will also allow us to enumerate conditions in length ω_1 (such as lines, planes, etc.).

To connect classical recursion to effective considerations, let $\leq_T$ denote the partial ordering of Turing degrees. Fix any reasonable coding[19] of $\mathbb{R}$ in 2^ω; if $x \in \mathbb{R}$, denote its code by $\overline{x} \in 2^\omega$.

Definition 2.3.3. *$X \subset 2^\omega$ is* **cofinal in the Turing degrees** *if it is cofinal in $\leq_T$. If $m \geq 1$ and $X \subset \mathbb{R}^m$ then X is cofinal in $\leq_T$ if $\{\overline{x} \,|\, x \in X\}$ is.*

We are now ready to present a special case[20] of Vidnyánszky's theorem. Let $\mathbb{D}$ denote the first quadrant of $\mathbb{R}^2$ (cf. section 6.3).

Theorem 2.3.4 ([152, Thm 1.3], $V{=}L$). *Let $F \subset \mathbb{D}^{\leq\omega} \times [0, \pi/2] \times \mathbb{D}$. If F is co-analytic and if for all $(A, p) \in \mathbb{D}^{\leq\omega} \times [0, \pi/2]$ the section*

$$F(A, p) = \{x \in \mathbb{D} \mid F(A, p, x)\}$$

is cofinal in $\leq_T$, then there exists a co-analytic set $X \subset \mathbb{D}$ compatible with F.

Note. As we must have cofinality for *all* pairs (A, p) (i.e. even when A is not a partial solution, or when p is already satisfied), we can isolate the following template for applying the theorem. For any pair (A, p) consider the cases:

1. *A is a partial solution and p is not yet satisfied.* We ensure that $\{x \,|\, A \cup \{x\}$ is a partial solution and x satisfies $p\}$ is cofinal in $\leq_T$.
2. *A is a partial solution but p is already satisfied.* We ensure that $\{x \,|\, A \cup \{x\}$ is a partial solution$\}$ is cofinal in $\leq_T$.

[19] We provide an explicit computable coding in section 6.2.

[20] Theorem 2.3.4 holds for all Polish spaces and all uncountable Borel subsets of an arbitrary Polish space that has a computable presentation [152].

3. *A is not a partial solution.* We may ignore both A and p—this case is usually trivial.

Vidnyánszky has hence recovered classical results [152], including the existence of a $\utilde{\mathbf{\Pi}}^1_1$ two-point set (Thm 5.2), a $\utilde{\mathbf{\Pi}}^1_1$ MAD (maximally almost disjoint) family of subsets of ω (Thm 5.1), and of a $\utilde{\mathbf{\Pi}}^1_1$ Hamel basis (Cor 4.11).[21] Beyond, Vidnyánszky and Andrea Medini have used Theorem 2.3.4 to classify separable metrisable topological spaces [115, Thm 12.2].

Proof (Sketch). Firstly, we note that cofinality in the hyperdegrees suffices for the result to hold. The constructed set will only contain self-constructible reals (cf. section 2.1.4). Firstly, augment F to obtain a set F' of tuples (c, A, p, x) where $(A, p, x) \in F$ and c codes a well-ordering of conditions which have already been satisfied (this is still $\utilde{\mathbf{\Pi}}^1_1$; we only pick $x \in \mathcal{C}_1$ such that $c \leq_h x$ and $L_{\omega_1^{(x)}}[x]$ agrees that c has coded all conditions so far with respect to $\leq_L$, and that p is the next condition to satisfy; this is possible since all sections are cofinal in the Turing degrees). Then uniformise to obtain single "solutions": each condition should be satisfied by an element. Not all such tuples are real solutions (some may not code "paths" through the "tree" of solutions coded by F), so we only keep those which are paths through F' (i.e. the history coded by c and A is correct). Finally, pick all x such that there is a partial solution and a history $(c, A, p) \leq_h x$ for which x is the unique solution. This is still $\utilde{\mathbf{\Pi}}^1_1$ by the Spector-Gandy theorem 2.1.23. $\square$

Remark. Why is cofinality in $\leq_h$ required? Since $x \in \mathcal{C}_1$, we know that $L_{\omega_1^{(x)}}[x] = L_{\omega_1^{(x)}}$, and so $L_{\omega_1^{(x)}}[x]$ is an initial segment of L itself. So if $L_{\omega_1^{(x)}}[x]$ thinks c is a history (of the first α reals with respect to $\leq_L$), then this is true

[21]The results on MAD families and Hamel bases are originally due to A. Miller [118].

in L by absoluteness (cf. Lemma 2.1.26). Given such an x, searching for a code of the correct history $c \leq_h x$ is $\underset{\sim}{\mathbf{\Pi}}^1_1$ by Spector-Gandy. Secondly, (why) we do we need the full strength of $V{=}L$? Some closure under constructibility is useful: if all reals are constructible, then by the end of the recursion we will actually have exhausted all conditions—hence the construction is completed in ω_1-many steps.

But is all of $V{=}L$ necessary? A proof analysis of Vidnyánszky's Theorem indicates that the statement "every real is constructible" *should* suffice for Theorem 2.3.4 to hold: the theorem relies fundamentally on the fact that (1) there are $\aleph_1$-many reals, and (2) the Spector-Gandy theorem holds, both of which follow if every real is constructible. A connection between Vidnyánszky's theorem and (degrees of) constructibility has already been identified: if Theorem 2.3.4 holds then every real is constructible [152, Thm 4.4]. The converse—whether "every real is constructible" implies Theorem 2.3.4—remains open, however, and leaves an avenue for future research [152, Problem 5.7].

Part I

κ-Computable Structure Theory

Chapter 3

Higher Computability Theory

We introduce **computable**, or **effective**, **structure theory**. After a brief history, we focus on developing higher computability theory. We then turn towards uncountable effective algebra. In chapter 4, we prove a novel result concerning the complexity of bases of uncountable free abelian groups, obtained in collaboration with Noam Greenberg, Saharon Shelah, and Daniel Turetsky. The paper upon which our results are based has been accepted for publication [55]; our work is principally structural instead of algebraic. Resources for the effective considerations we describe include [135, 54, 53]. Comprehensive sources on computable structure theory are [2, 122]; for a historical survey (including a rich bibliography) we recommend [41], from which we extract a brief outline.

Computable structure theory tries to measure the complexity of mathematical structures. Constructing structures via computability-theoretical means (hence measuring the effective strength required to do so) caught the attention of researchers independently: in the mid 1970s, the Russian school around Ershov, Goncharov, and Nurtazin [33, 51, 125] developed the notion of *constructivisation* at Novosibirsk [2, 0.2]. At the same time, Metakides and Nerode, among others in the United States, investigated the effective content of classical mathematics, in particular in abstract and linear algebra [116, 117]. These developments have since converged, rendering computable structure theory a vibrant research area.

The computability-theoretical properties of classical mathematical structures are normally investigated via **reductions** on the Turing degrees. This is a common approach in logic: take a group, ring, vector space, or topological space for example, and embed—or *encode*—it in some space that admits notions of *classification*. These **presentations**, or **copies**, are then studied effectively; e.g. a **computable ω-presentation** of a structure $\mathcal{M}$ is a structure whose domain is a computable subset of ω, and whose functions, relations, and constants are *uniformly* Turing computable [122, 1.1.1]. Avenues of contemporary computable structure theory include:

- *computable categoricity*: the study of those theories all of whose computable ω-copies are computably isomorphic. Classifying necessary and sufficient conditions for computably categoricity has been an active area of research for decades [122, 8.2, Table 2]—however, *generally* there is no simple characterisation of computable categoricity [28]: the authors' result is in style similar to the theorem on identifying uncountable free abelian groups [56], cf. section 3.2.1: relative to a suitable context, that there is no natural characterisation bar the "obvious" one.
- *degree spectra*: the collection of sets computing a presentation for a given structure. Formally, the *spectrum of a structure* $\mathcal{M}$ is given by:

  $$\mathrm{Sp}(\mathcal{M}) = \{X \in 2^\omega \mid X \text{ computes an } \omega\text{-copy for } \mathcal{M}\}$$

  Clearly, the degree spectrum is upwards closed in the Turing degrees.

More interestingly, if $\mathcal{M}$ is non-trivial[1] and X computes an ω-copy for $\mathcal{M}$ then there exists an ω-copy for $\mathcal{M}$ of Turing degree[2] X; this is *Knight's theorem* [75] (cf. [122, Thm 1.2.1]), which implies

$$\mathrm{Sp}(\mathcal{M}) = \{X \in 2^\omega \mid (\exists M)(M \text{ is an } \omega\text{-copy of } \mathcal{M} \text{ and } M \equiv_T X)\}.$$

Hence, model-theoretically isomorphic structures may have very different effective contents, admitting a finer classification of structures than classical model theory. Realising sets of degrees as spectra of structures forms a major research theme [122, Chapter 5] (cf. Question 4.5.1).

Tools in computable structure theory range from model-theoretical (using e.g. atomic diagrams and infinitary languages, particularly $\mathcal{L}_{\omega_1\omega}$; cf. Scott sentences, and Scott's isomorphism theorem [137]) to computability-theoretical (e.g. forcing and hyperarithmetic theory [2]).

Its ideas are not restricted to ω: below, we outline how to define computability theory on uncountable cardinals, and that the resultant theory behaves similarly to classical ω-Turing computability[3], enabling us to investigate *uncountable* computable structure theory.

3.1 Computability in the Transfinite

Traditionally, the domain of discourse of computability theory comprises the natural numbers. There exists a straightforward recasting of the theory into the set-theoretical universe, with the collection of **hereditarily finite** sets

[1] $\mathcal{M}$ is *not trivial* if there exists no finite tuple so that every permutation of the domain of the structures that fixes that tuple is actually an automorphism on the whole structure.

[2] Recall that every ω-presentation is a subset of ω, hence has a Turing degree.

[3] Results from algebra and order theory can be transferred onto ω_1 [54], e.g. every ω_1-computable vector space over a countable field has a ω_1-computable basis [54, Prop 4.1].

HF serving as the new domain. This set serves as a blueprint for a general class of sets which naturally extend computability theory into the transfinite: the **admissible sets**. We outline the construction of HF below, foreshadowing the ideas that follow in section 3.1.1.

Recall a set x is **transitive** if whenever $y \in x$ then $y \subset x$. Hence transitivity fails if there exists $y \in x$ such that for some $z \in y$ we have $z \notin x$; in terms of graphs, there is a path from x to y and one from y to z but none from x to z. This can be resolved by "adding in" all possible paths inductively[4]: for any x define by recursion

$$x_0 = x \quad \text{and} \quad x_{n+1} = \bigcup x_n.$$

Then let $\mathrm{tc}(x) = \bigcup\{x_n \,|\, n < \omega\}$.

Lemma 3.1.1. *If x is a set then $\mathrm{tc}(x)$ is the smallest (with respect to inclusion) transitive set containing x.*

Hence we define the hereditarily finite sets by

$$\mathsf{HF} = \{x \,|\, |\mathrm{tc}(x)| < \aleph_0\}.$$

HF permits an even simpler characterisation. In the following, if M is a set and $x \in M$ we write $\varphi^M(x)$ to mean $M \models \varphi(x)$.[5]

Lemma 3.1.2. $\mathsf{HF} = V_\omega = L_\omega$

Proof. It is easy to see that $V_\omega = L_\omega$; for the non-trivial direction, it suffices to show that $L_n = V_n$ for all n. Suppose the nontrivial inclusion $L_m \subset V_m$

[4]More visually, for every x consider the graph $G(x)$ whose vertices are y such that there exist $n < \omega$ and $x_1, \ldots, x_n$ such that $y \in x_1 \in \ldots \in x_n \in x$. Then $\mathrm{tc}(x) = \mathrm{field}(G(x))$.

[5]This identification is not valid for proper classes by the usual undefinability-of-truth and incompleteness arguments of Tarski and Gödel.

holds below $n+1$. If $x \in V_{n+1}$ then $x = \{x_0, \ldots, x_k\}$ for some $k < \omega$ and each $x_i \in V_n = L_n$; hence $\{y \mid \varphi^{L_n}(y, x_0, \ldots, x_k)\}$ where

$$\varphi(y, x_0, \ldots, x_k) \text{ is } y = x_0 \vee \ldots \vee y = x_k$$

defines x in L_n, and thus $x \in L_{n+1}$. Similarly, it is also easily seen that $V_\omega \subset \mathsf{HF}$: this follows from the fact that $|V_n| = 2^n < \aleph_0$, and hence $|\operatorname{tc}(x)|$ is also finite. The other direction follows from Lemma 3.1.5 below. □

Classical computability theory can now be recast inside HF via coding:

Theorem 3.1.3 ([4, II.2.4])**.** *There exists a computable bijection $f\colon \omega \to \mathsf{HF}$.*

The structure HF is also known as $H(\aleph_0)$. This notation alludes to a more general class of sets.

Definition 3.1.4. *For every cardinal κ, we define $H(\kappa) = \{x \mid |\operatorname{tc}(x)| < \kappa\}$.*

The sets $H(\kappa)$ play an important role in axiomatic set theory; for instance, for any regular uncountable κ the set $H(\kappa)$ satisfies $\mathsf{ZFC} - \mathsf{Power\ Set}$, assuming the universe satisfies AC [80, IV.6.5]. Further, $H(\kappa)$ is a model of ZFC proper if κ is strongly inaccessible; in that case we also have $H(\kappa) = V_\kappa$ [80, IV.6.6]. If κ is infinite we already obtain a somewhat weaker result.

Lemma 3.1.5 ([80, IV.6.2])**.** *If κ is an infinite cardinal then $H(\kappa) \subset V_\kappa$.*

Lemma 3.1.6 ([80, IV.6.4])**.** *Each $H(\kappa)$ is transitive. If x is transitive and $|x| < \kappa$ then $x \in H(\kappa)$. Further, $H(\kappa) \cap \mathrm{ON} = \kappa$.*

3.1.1 Admissibility Theory

We have made an intuitive case for how classical Turing computability theory can be transferred into set theory via HF. Naturally, the set-theoretical universe is rich, so there is hope that higher cardinality analogues to HF providing a domain for computation exist. We observed that transitivity of our domain is desirable, and that closure under all reasonable operations is a must. Since $\mathsf{HF} = V_\omega = L_\omega$, it is hence tempting to consider levels of the cumulative hierarchies V and L, and in particular of the ordinals which yield a suitable domain—this is the study of **admissibility theory**, and such ordinals are called **admissible**. This study is introduced carefully in [25] (in the context of L) as well as in [4] (in terms of definability).

Finding a suitable domain for transfinite computability theory is one thing, yet developing the recursion theory itself is another. This is what we tackle after, when we consider **α-recursion theory** in section 3.1.2. For an introduction to ordinal recursion theory see [16, 52, 135]. Another well-written (yet brief) introduction is given in [53].

Definition 3.1.7. *A set M is* **amenable** *if it satisfies the following: it is transitive; it satisfies* Pairing *and* Union*; $\omega \in M$; the axiom* Cartesian*, which says $(\forall x, y \in M)(x \times y \in M)$; and $\Delta_0(M)$-*Comprehension*: if $R \subset M$ is $\Delta_0(M)$ (i.e. with parameters), then $(\forall x \in M)(x \cap R \in M)$.*

It is easily checked that HF is amenable. However, the axioms of amenability do not suffice to provide a sufficiently strong theory to model computability. For instance, while rank-analysis shows that V_α is amenable if α is a limit, it is not necessarily closed under addition; e.g. set $\alpha = \omega \cdot 2$ (this was already pointed out in [53]). Since all domains of computability should be closed under all "computable" operations, this is rather undesirable. Admissibility provides the required strengthening via definability.

Definition 3.1.8. *A set M is* **admissible** *if it is amenable and also satisfies*

$\Delta_0(M)$-Collection*, which states: if* $R \subset M^2$ *is* $\Delta_0(M)$ *satisfies*

$$(\forall x \in M)(\exists y \in M)((x, y) \in R)$$

then for any $x \in M$ *(the* domain*) there exists* u *(the* collecting set*) which, for each* $y \in x$ *(the* input*) contains* z *(an* output*) such that* $R(y) = z$*:*

$$(\forall x \in M)(\exists u \in M)(\forall y \in x)(\exists z \in u)((y, z) \in R).$$

The theory of admissible sets admits an axiomatisation via **Kripke-Platek set theory** KP, the (strictly weaker) subtheory of ZF due to Saul Kripke and Richard Platek [79, 129] given by the following axioms [135]: Pairing; Union; $\Delta_0(M)$-Comprehension; $\Delta_0(M)$-Collection; and Infinity. Thus:

Lemma 3.1.9. *A set* M *is admissible iff* M *is transitive and satisfies* KP*.*

Suppose M is admissible and that $X \subset M$. If X is $\Delta_1(M)$ (i.e. we think of X as being computable), then X should be approximable by elements of M: if $x \in M$ then the "initial segment" $x \cap X$ should also be an element of M. The following lemma establishes this.

Lemma 3.1.10 ([25, 11.3, 11.4])**.** *Every admissible set* M *satisfies* $\Delta_1(M)$-Comprehension *and* $\Sigma_1(M)$-Collection*.*

Where can we find admissible sets? The blueprint example is easily defined: the natural extension of HF to the transfinite does the trick:

Theorem 3.1.11 ([25, I.11.2],[4, II.3.1])**.** *For every uncountable* κ *the set* $H(\kappa)$ *is an admissible set.*

We now work towards identifying admissible sets in the cumulative hierarchy. As we shall see, the constructible hierarchy yields desirable structure.

Definition 3.1.12. *An ordinal α is* **admissible** *if L_α is an admissible set.*

Admissible ordinals are not rare. Firstly, if $V{=}L$ then we automatically obtain a large number of admissibles for free, via Theorem 3.1.11:

Lemma 3.1.13 ($V{=}L$)**.** *$H(\kappa) = L_\kappa$ for every infinite cardinal κ.*

Proof. If $x \in L_\kappa$ then $x \in L_{\alpha+1}$, and so $x \subset L_\alpha$ for some $\alpha < \kappa$. Observe that if $x \subset L_\alpha$ then $\mathrm{tc}(x) \subset L_\alpha$: for every $y \in \mathrm{tc}(x)$ there exist $n < \omega$ and $x_1, \ldots, x_n$ such that $y \in x_1 \in \ldots \in x_n \in x$, and thus $\mathrm{rank}(y) < \mathrm{rank}(x) = \alpha$. (Here, by rank we mean the rank in the L-hierarchy.) But now recall that $|L_\alpha| = |\alpha| < \kappa$. Hence $|\,\mathrm{tc}(x)| \leq |\alpha| < \kappa$, as needed.

For the other direction, assume $x \in H(\kappa)$. Since $V{=}L$, we must have $x \in L_\delta$ for some limit ordinal δ. Take an elementary substructure X of L_δ of cardinality $\lambda = |\,\mathrm{tc}(x)|$ for which $x \in X$ and $\mathrm{tc}(x) \subset X$. By assumption, $\lambda < \kappa$. Now, the condensation lemma tells us that $X = L_\alpha$ for some α; since $|X| = \lambda < \kappa$, we must have $\alpha < \kappa$. Since $\mathrm{tc}(x)$ is transitive, it is preserved under the collapsing map, and thus so is x. Therefore $x \in L_\alpha \subset L_\kappa$. □

The following lemma allows us to characterise admissible ordinals.

Lemma 3.1.14 ([25, Lem II.7.1])**.** *An ordinal α is admissible if and only if there exists an admissible set M such that $M \cap \mathrm{ON} = \alpha$.*

We can now show that there are many admissible ordinals even if $V{\neq}L$.

Lemma 3.1.15. *Every uncountable cardinal is admissible. Every α-stable ordinal is admissible. If κ is an uncountable cardinal then there are κ-many admissible ordinals below κ.*

The second claim is [135, VII.2.8]; the third is mentioned in [25, II.7] without proof. Recall that if α, β are ordinals then β is **α-stable** if $\beta \leq \alpha$ and L_β is a 1-elementary substructure of L_α [4, V.7.3].

Proof. If $V{=}L$, then Theorem 3.1.11 and lemma 3.1.13 imply the first part. Otherwise Theorem 3.1.11 and lemma 3.1.14 suffice. The second part is proven in [4, V.7.5].

Suppose κ is an uncountable cardinal. By the first part, it is admissible. Observe that admissibility is a first-order property, as it is characterised by KP. If $\lambda^+ = \kappa$, use Skolem-Löwenheim and condensation to obtain an elementary substructure L_{α_0} of L_κ of cardinality λ. By first-orderedness of admissibility, α_0 is an admissible ordinal itself. If α_β exists, take an elementary substructure M_β of L_κ, again of cardinality λ, but this time require that $\alpha_\beta + 1$ is a subset of M_β. By condensation, we obtain an elementary substructure $L_{\beta'} \subset L_\kappa$ for which $\beta' > \alpha_\beta$ (since it contains $\alpha_\beta + 1$, again preserved by the Mostowski collapse). Hence, put $\alpha_{\beta+1} = \beta'$. If β is a limit ordinal, carry out the same construction on $\bigcup_{\gamma<\beta} \alpha_\gamma$. Clearly, each α_β is admissible, and regularity of κ yields the result. If κ is a limit cardinal, we go through the same argument; between any two successors κ and κ^+, carry out the same construction as before; this yields κ^+-many admissibles. At limit stages, collect all the admissibles. $\square$

There is one more technical property we need, which is motivated by oracles. Consider computability on the natural numbers, and let $A \subset \mathsf{HF}$ be an oracle. Then $A \cap x \in \mathsf{HF}$ for all $x \in \mathsf{HF}$; we call A *regular*[6] *for* HF.

[6]Regularity is essential to carry out priority constructions; see e.g. [97]. Also cf. the Introduction of [54]

Unfortunately, if $\alpha > \omega$ then not every $A \subset L_\alpha$ is necessarily regular for L_α. This can be fixed as follows: under $V{=}L$, every $A \subset L_\alpha$ is regular for L_α provided α is an infinite cardinal [15, p. 9]. Conversely, if $V{=}L$ and α is not a cardinal, then there exists $A \subset L_\alpha$ and $x \in L_\alpha$ for which $A \cap x \notin L_\alpha$ [97].

Lemma 3.1.16 ($V{=}L$)**.** *Every $A \subset L_\kappa$ is regular for L_κ iff κ is a cardinal.*

Combining the lemma with Theorem 3.1.11 and Lemma 3.1.13 implies:

Fact. Assuming $V{=}L$, if α is a cardinal then L_α is a suitable domain for higher computability theory.

3.1.2 Working in L_κ

From now on, let κ be a regular cardinal, and assume $V{=}L$. We introduce computability theory on L_κ, or **κ-computability theory**. We closely follow Greenberg and Knight's account [54], who give a comprehensive account of the development of L_{ω_1}, assuming every real is constructible—this can be seen as a natural analogue of $\mathsf{HF} = L_\omega$ via Lemma 3.1.13: every real is constructible if and only if $L_{\omega_1} = H(\aleph_1)$ [54].

To transfer classical ideas (such as Post's theorem) to higher settings, we argue via definability.[7] Our language is the **first-order language of set theory**, denoted by $\mathcal{L}$, whose non-logical symbols are $=$ and $\in$. We arrange its formulas in the **Lévy hierarchy** [85]: a formula is Δ^0_0 if it is of the form

$$(Q_1 x_1 \in y_1) \cdots (Q_n x_n \in y_n) \varphi(x_1, y_1, \ldots, x_n, y_n)$$

where $Q_i \in \{\exists, \forall\}$ and φ is a boolean combination of atomic formulas. These formulas correspond obviously to a subclass of computable sets: the search

[7] A brief but very readable summary of some of the results below in a general set-theoretical context can be found in Chapter 0 of [68].

for witnesses (or counterexamples, depending on whether $Q = \exists$ or $\forall$) is bounded, hence can be carried out in κ-finite time. By induction, a formula is Σ^0_{n+1} if it is of the form

$$(\exists x_1)\cdots(\exists x_n)\varphi(x_1,\ldots,x_n)$$

where φ is Π^0_n, and similarly, a formula is Π^0_{n+1} if it is of the form

$$(\forall x_1)\cdots(\forall x_n)\psi(x_1,\ldots,x_n)$$

where ψ is Σ^0_n. As our work is constrained to certain sets (for instance L_κ), the construction of the hierarchy to this point suffices. We will always argue semantically to determine the classification of formulas inside the structure we are working in; most of the time, this will be L_κ for a regular cardinal κ.

To complete the formalisation, we recall that natural numbers are trivially computable in the classical context. This hints at the fact that we ought to extend our formulas to allow parameters, which yields:

Definition 3.1.17. *A formula is $\Sigma^0_n(X)$ if it is Σ^0_n in the language augmented by all symbols of X as constants—these constants are interpreted in the obvious way. Similar definitions apply to $\Pi^0_n(X)$ and $\Delta^0_n(X)$. We denote the resultant language by $\mathcal{L}(X)$.*

Definition 3.1.18. *A set $X \subset L_\kappa$ is* **κ-computably enumerable** *if there exists a $\Sigma^0_1(L_\kappa)$-formula defining X in L_κ. Similarly, X is* **κ-computable** *if both X and its complement $L_\kappa \setminus X$ are κ-c.e.*

Model-theoretically, a set $X \subset L_\kappa$ is κ-c.e. if and only if there exists a $\Sigma^0_1(L_\kappa)$-formula φ of one free variable such that

$$a \in X \iff (L_\kappa, \in) \vDash \varphi(a).$$

For instance, the set of ordinals below κ is κ-computable: $x \in L_\kappa$ is an ordinal if and only if x is transitive and linearly ordered by $\in$; this is Δ_0^0:

$$(\forall y \in x)(\forall z \in y)(z \in x) \wedge (\forall y \in x)(\forall z \in x)((y \neq z) \to (y \in z \vee z \in y))$$

Definition 3.1.19. *A function $f\colon L_\kappa \to L_\kappa$ is called* κ**-computable** *if its domain is κ-computable and its graph is κ-c.e. If the graph of f is κ-c.e. but its domain is not κ-computable, we call f* κ**-partial computable**.

Since admissible sets are closed under all computable operations, we can avoid coding. This simplifies many classical proofs (cf. [54, Lem 2.3]). The proof of the **transfinite computable recursion theorem**, for instance, follows exactly the classical proof (e.g. [63, Thm 2.15] and [80, I.9.3]), while keeping track of quantifiers:

Theorem 3.1.20 ([54, Prop 2.4])**.** *If $I\colon L_\kappa \to L_\kappa$ is κ-computable, then there exists a unique κ-computable $f\colon \kappa \to L_\kappa$ such that for all $\alpha < \kappa$,*

$$f(\alpha) = I(f \restriction \alpha).$$

An important feature of classical Turing computability is **linearisation**: recall from Theorem 3.1.3 that there exists a (classically) computable bijection $f\colon \omega \to \mathsf{HF}$. This can be extended to L_κ:

Theorem 3.1.21 ([54, Cor 2.6])**.** *There is a κ-computable bijection $f\colon \kappa \to L_\kappa$.*

The proof uses the well-ordering $<_L$: since $<_L$ is an end-extension from L_α to $L_{\alpha+1}$ for every $\alpha < \kappa$ (in fact for every $\alpha \in \mathrm{ON}$), asking where exactly $x \in L_\kappa$ sits in $<_L$ is c.e. and co-c.e. since $<_L$ is total. We now show that the structure of κ-Turing degrees behaves as expected. In particular, the usual hardness results hold[8]:

[8]The satisfaction relation on L_α for $\alpha < \kappa$ is computable; and every φ in $\Sigma_1^0(L_\kappa)$ is upwards absolute, its bounded-quantifier part even absolute for transitive classes.

Lemma 3.1.22 ([54, 2.2])**.** *There exists a universal κ-c.e. set W.*

Corollary 3.1.23 ([54, Prop 2.10])**.** *W is not κ-computable.*

We state without proof the extensions to κ-computability of the following classical theorems (all can be found in [54, 2.1-2.3]): a set is κ-computable if and only if so is its characteristic function; there exists an effective listing of all κ-p.c. functions; there exists a uniform enumeration of all κ-c.e. sets; the s-m-n-theorem; and Kleene's recursion theorem. One can also show that a non-empty set is κ-c.e. if and only if it is the domain of a κ-p.c. function if and only if it is the range of a total κ-computable function. These follow since L is computably well-ordered, and from linearisation and recursion on κ.

Next, we talk about relativisation and Turing reductions. In the classical context, an oracle B invites questions and always gives the correct answer. Hence, a natural transfer into definability theory should allow B as a parameter in the language. We will consider this formal connection between extending the language by predicates in more detail in section 3.1.3. For now, let us just say that $\Sigma^0_1(L_\kappa; B)$ formulas are constructed as expected: they are $\Sigma^0_1(L_\kappa)$ formulas in the language of set theory extended by an additional symbol "B", interpreted in the obvious way, denoting $B \subset L_\kappa$.

A second classical formalisation of Turing reductions uses functionals.

Definition 3.1.24. *An* **enumeration functional** *is a set of pairs $(\sigma, x) \in 2^{<\kappa} \times L_\kappa$.*

We think of κ-finite strings as **conditions**: if Φ is an enumeration functional and $(\sigma, x) \in \Phi$ then any subset of κ that interprets Φ and contains σ must contain x. Thus, Φ acts on subsets of κ; if $B \subset \kappa$, the resultant set Φ^B is determined by the initial segments of B:

$$x \in \Phi^B \iff (\exists \sigma \prec B)((\sigma, x) \in \Phi).$$

Enumeration functionals and definability yield the same notion of reduction:

Theorem 3.1.25 ([54, Prop 2.14])**.** *Suppose $A \subset L_\kappa$ and $B \subset \kappa$. Then there exists a κ-c.e. enumeration functional Φ such that $A = \Phi^B$ if and only if A is $\Sigma_1^0(L_\kappa; B)$. We say A* **is κ-c.e. relative to** *B.*

Definition 3.1.26. *A set $A \subset L_\kappa$ is* **κ-computable in** *B if A is both κ-c.e. in B and κ-co-c.e. in B. We write $A \leq_\kappa B$.*

All the results we listed earlier relativise via the obvious definability approaches: there exists a universal B-c.e. set; an effective enumeration of all B-c.e. partial functions; as well as a B-halting set [54, 2.4].

Another classical approach to Turing reductions uses Turing functionals; these perform the same role in L_κ.

Definition 3.1.27. *A* **Turing functional** *Φ is a downwards closed consistent set of pairs $(p, q) \in 2^{<\kappa} \times 2^{<\kappa}$:*

- *If $(p, q), (p', q') \in \Phi$ and p, p' are comparable, then so are q, q'.*
- *If $(p, q) \in \Phi$ and $q' \prec q$ then $(p, q') \in \Phi$.*

Every functional acts on subsets of κ: if $B \subset \kappa$ then

$$q \prec \Phi^B \iff (\exists p \prec B)((p, q) \in \Phi).$$

We also denote this induced function by F_Φ.

While enumeration functionals characterise κ-c.e.-reductions, Turing functionals characterise the κ-Turing reduction:

Lemma 3.1.28 ([54, Prop 2.16].)**.** *$A \leq_\kappa B$ if and only if there exists a κ-c.e. Turing functional Φ for which $A = \Phi^B$.*

As a corollary, the relation $\leq_\kappa$ is a well-behaved hierarchy[9].

Theorem 3.1.29 ([54, Prop 2.17])**.** *$\leq_\kappa$ is an equivalence relation.*

3.1.3 Classes in the Definable Context

The principal viewpoint of our investigation of transfinite computability uses definability. As we are working with effective notions, all elements of L_κ are trivially computable; to develop an interesting theory, we focussed on subsets of L_κ. In this section, we consider collections of subsets of κ. In classical parlance, these can be considered **classes** of **κ-reals** (total functions from κ to 2, each coding a subset of κ). This investigation is formalised in the language of first-order logic $\mathcal{L}(L_\kappa)$. However, we augment our language by second-order, i.e. subset, parameters. In the following, φ is always a formula in $\mathcal{L}(L_\kappa)$. First-order quantifiers quantify over[10] L_κ, while second-order quantifiers quantify over *subsets* of L_κ. As a convention, the letters X, Y usually denote variables denoting subsets of κ.

We will also consider relativised notions: if $B \subset \kappa$ we can enrich the language to allow B as a fixed predicate; the resultant first-order language of set theory is denoted by $\mathcal{L}(L_\kappa; B)$.

Definition 3.1.30. *Let $\varphi(X)$ be a formula in $\mathcal{L}(L_\kappa)$. A class $R \subset 2^\kappa$ is* **definable from** φ *if*

$$A \in R \iff (L_\kappa, \in, (A)) \models \varphi(A).$$

Definition 3.1.31. *A class $R \subset 2^\kappa$ is a* **Σ^1_1-class** *if there exists a formula $(\exists Y)(\varphi(Y, X))$ in $\mathcal{L}(L_\kappa)$ such that*

[9] An arithmetical hierarchy for L_{ω_1} has been introduced in [11].

[10] Recall from Theorem 3.1.21 that we may pass between κ and L_κ computably.

$$A \in R \iff (L_\kappa, \in, (B, A)) \models \varphi(B, A) \text{ for some } B \subset \kappa.$$

Definition 3.1.32. *Suppose $D \subset \kappa$. A class $R \subset 2^\kappa$ is a $\Sigma^1_1(D)$-**class** if there exists a formula $(\exists Y)(\varphi(Y, X))$ in $\mathcal{L}(L_\kappa; D)$ such that*

$$A \in R \iff (L_\kappa, \in, (B, A); D) \models \varphi(B, A) \text{ for some } B \subset \kappa.$$

*We then say that R is a $\underset{\sim}{\mathbf{\Sigma}}^1_1$-**class**.*

To talk about reductions between classes, we need means to define the complexity of functions between κ-reals, not just between κ-finite elements as in Definition 3.1.19. Such a notion can be developed via Turing functionals (cf. Definition 3.1.27). In fact, Turing functionals characterise exactly the partial continuous functions under the topology whose basic open sets are determined by conditions $p \in 2^{<\kappa}$: each p determines $[p] = \{f \in 2^\kappa \mid p \prec f\}$.

Lemma 3.1.33. *Let $F \colon 2^\kappa \to 2^\kappa$. Then F is partial continuous if and only if $F = F_\Phi$ for some Turing functional Φ.*

Proof. If F is partial continuous, define $(p, q) \in \Phi$ iff $[p] \cap \operatorname{dom}(F) \subset F^{-1}[q]$. Then $q \prec \Phi^A \iff (\exists p \prec A)((p, q) \in \Phi) \iff (\exists p)(A \in [p] \wedge [p] \subset F^{-1}[q]) \iff q \prec F(A)$. Conversely, for Φ a functional, $F_\Phi^{-1}[q] = \{B \in \operatorname{dom}(F_\Phi) \mid q \prec \Phi^B\} = \{B \in \operatorname{dom}(F_\Phi) \mid (\exists p \prec B)((p, q) \in \Phi)\} = \bigcup\{[p] \cap \operatorname{dom}(F_\Phi) \mid (p, q) \in \Phi\}$, which is open in $\operatorname{dom}(F_\Phi)$. $\square$

The complexity of the induced function is determined by the complexity of its functional: since Turing functionals are subsets of L_κ, their recursive properties can be investigated *inside* the admissible structure L_κ. This gives rise to a natural extension of Definition 3.1.19:

Definition 3.1.34. *A function $F \colon 2^\kappa \to 2^\kappa$ is κ-**partial computable** if its Turing functional $\Phi \subset L_\kappa$ is κ-c.e. If F is also total, it is κ-**computable**.*

To construct κ-computable functions between κ-reals, we argue ad hoc: $F\colon 2^\kappa \to 2^\kappa$ is κ-computable if and only if there exists a κ-computable process in the sense of Definition 3.1.26 that, *uniformly* on input[11] A, correctly computes (any initial segment of) $F(A)$ in κ-finite time. This approach is used implicitly to prove [56, Thm 3.1]. Finally, we turn towards reductions and completeness.

Definition 3.1.35. *Suppose $R, S \subset 2^\kappa$ are classes. Then S* **is reducible to** *R if there exists a κ-computable function $G\colon 2^\kappa \to 2^\kappa$ such that*

$$A \in S \iff G(A) \in R$$

for all $A \in 2^\kappa$. The map G is called a **reduction**. *Let Γ be a collection of classes. A class R is* **Γ-complete** *if $R \in \Gamma$ and if every $S \in \Gamma$ is reducible to R: for every $S \in \Gamma$ there exists a κ-computable reduction G_S for which*

$$A \in S \iff G_S(A) \in R.$$

3.2 Characterising Free Abelian Groups

The results we exposé in this section are based on an investigation of uncountable free abelian groups due to Greenberg, Turetsky, and Westrick [56]. The question they ask is the following: with (the multiplication table of) an uncountable free abelian group G in hand, how easy is computing a basis for it? This investigation is tightly related to characterising free abelian groups:

Definition 3.2.1. *A group $(G, *)$ is* **free abelian** *if it has a* **basis***: a linearly independent subset which spans it.*

[11] "on input A" means we have oracle access to A.

For more details on free abelian groups see e.g. [45]. We give a basic characterisation: for κ a cardinal, let $\mathbb{Z}^\kappa$ be the group of formal sums

$$\sum_{\beta<\kappa} n_\beta \mathbf{e}_\beta$$

where each $n_\beta \in \mathbb{Z}$ and each $\mathbf{e}_\beta$ is a dedicated basis element from the β-th copy of $\mathbb{Z}^\kappa$, yet only finitely many n_β are non-zero. The associated group operation is given by coordinate-wise addition: let $\mathbf{e}_\alpha \in 2^\kappa$ be such that

$$\mathbf{e}_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{Z}^\kappa$ is free abelian. For instance $(1, 2, 3, 0, \dots) \in \mathbb{Z}^\kappa$ can be written as

$$(1, 2, 3, 0, \dots) = \mathbf{e}_0 + 2\mathbf{e}_1 + 3\mathbf{e}_2.$$

Generally, if B is a set of dedicated basis elements, let $\mathbb{Z}^{(B)}$ denote the according set of formal sums. The following isomorphism is hence established:

$$\mathbb{Z}^{(B)} \cong \bigoplus_{b\in B} b\,\mathbb{Z}\,.$$

Lemma 3.2.2. *For every cardinal κ the group $\mathbb{Z}^\kappa$ is free abelian. Every free abelian group is isomorphic to $\mathbb{Z}^\kappa$ for some cardinal κ. If B, B' both have cardinality then $\mathbb{Z}^{(B)}$ is isomorphic to $\mathbb{Z}^{(B')}$.*

Proof. Let $B = \{\mathbf{e}_\alpha \,|\, \alpha < \kappa\}$; it is easily seen that B is a basis for $\mathbb{Z}^\kappa$. If G is free abelian, let B be a basis with $|B| = \kappa$. Then we can well-order $B = \{b_\alpha \,|\, \alpha < \kappa\}$, and so define a map φ mapping $b_\alpha \mapsto \mathbf{e}_\alpha$, which extends linearly to a group homomorphism from G to $\mathbb{Z}^\kappa$. The third claim follows. $\square$

By cardinality considerations, $\mathbb{Z}^\kappa$ has order $\aleph_0 \cdot \kappa$, and if $\kappa \geq \aleph_0$ then any basis for $\mathbb{Z}^\kappa$ has cardinality κ.

3.2.1 Extending Independent Sets to Bases

It is easily seen that if G is free abelian with basis B then every $g \in G$ has a unique representation of of basis elements. On the other hand, constructing bases for free abelian groups recursively is complicated. Below, we explain why; we stick closely with the presentation of [56].

When considering bases, vector spaces come to mind. Bases for vector spaces can be constructed recursively by extension. The determining property is linear independence: if V is a vector space and $B \subset V$ is linearly independent and does not span V, then we can extend B to some B' by adjoining $x \in V \setminus \mathrm{span}(B)$ to B. This process is easily shown to work for vector spaces of all cardinalities (provided AC [9]). However, it does not necessarily work for free abelian groups, with problems arising at the limit stage.

Unlike vector spaces, free abelian groups contain maximal linearly independent subsets which are *not* bases: each $\{n\} \subset \mathbb{Z}$ is linearly independent, and maximally so by Bézout's lemma. However, only $\{1\}$ and $\{-1\}$ generate $\mathbb{Z}$. Hence, linear independence is not strong enough to characterise the step-by-step construction of bases.

This imbalance can be overcome via a stronger notion of independence than linear independence, originally due to Lev Pontryagin [130] and called **P-independence** in [56]—its generalisation as an extension of the algebraic idea of p-independence (see [45, Chapter 5, Section 5.5] for details) is due to Downey and Melnikov [26]. The lemma allowing the recursive basis construction to continue is the following, restated in the language of [56]:

Lemma 3.2.3 ([26, Prop 4.5])**.** *If $B \subset \mathbb{Z}^\omega$ is finite P-independent, then for every $g \in \mathbb{Z}^\omega \setminus \mathrm{span}(B)$ there is a finite P-independent $B' \supset B$ containing g.*

Hence, the issue of constructing bases recursively *provided* the partial bases are finite can be overcome by using P-independence in place of linear independence, showing that for every countable free abelian group a basis can be constructed recursively. This, however, does not solve the general issue of extending partial bases into the uncountable—we only sidestepped it in the countable case by introducing a stronger notion of independence and ensuring the construction has length ω. Indeed, it can be shown that P-independence is also not strong enough to ensure *uniform* closure under limit step extensions [56, Ex 1.3].

So the general question remains: is there a form of "strong" independence, certainly stronger than linear independence and P-independence, which ensures recursive constructions of bases of arbitrary free abelian groups succeed? To answer this, we need a general framework which classifies acceptable notions of independence "strength". Given the theme of this thesis, it should not be surprising that definability in the context of computability theory fits the bill. If an algorithm existed that upon input of the multiplication table of a group computes a basis if and only if the group is free abelian, then that would provide us with such a strong notion of independence. However, as Greenberg, Turetsky, and Westrick proved, no such notion exists: classifying free abelian groups is as difficult as possible [56].

To explain their theorems, we return to computability considerations. Suppose $G \subset \kappa$ is a κ-computable group; that means, the domain and the graph of $(G, *)$, i.e. the set of triples (g, h, k) for which $g * h = k$ (where $*$ denotes the group operation) are κ-computable as subsets of L_κ. To identify whether G is free abelian, it suffices to search over all subsets of κ for a basis of G. This is clearly a Σ^1_1 statement in $\mathcal{L}(L_\kappa)$, since the formula expressing

"X is a basis for G" with X being a free variable in the language $\mathcal{L}(L_\kappa)$ is definable without second-order quantification: assuming the group is additive, the (abbreviated) arithmetical formula—denoted by $\psi(G, X)$—that is the conjunction of

$$(\forall x \in G)(\exists n < \omega)(\exists f \colon n \to \mathbb{Z})(\exists h \colon n \to X) \left(x = \sum_{i<n} f(i)h(i) \right) \tag{3.1}$$

$$(\forall n)(\forall f \colon n \to \mathbb{Z})(\forall h \colon n \to X) \\ \left(0 = \sum_{i<n} f(i)h(i) \to (\forall m < n)(f(m) = 0) \right) \tag{3.2}$$

and the formula "$X \subset G$" does the trick. Hence let $\mathcal{G}(\kappa)$ denote the class of free abelian groups of order κ. Definition 3.1.31 now implies:

Lemma 3.2.4. *The class $\mathcal{G}(\kappa)$ is a Σ^1_1-class in 2^κ.*

Proof. The axioms of group theory only quantify over elements, so their concatenation with the arithmetical part of $\psi(G, X)$ remains first-order—denote this new formula by ψ^*. Thus, $(\exists B)(\psi^*(G, B))$ is Σ^1_1, and so

$$G \in \mathcal{G}(\kappa) \iff (L_\kappa, \in, (B, G)) \vDash \psi^*(G, B) \text{ for some } B \subset \kappa. \qquad \square$$

This begs the question: is there a pointclass Γ *simpler* than Σ^1_1 which defines $\mathcal{G}(\kappa)$? Such a characterisation would yield an algorithm we could apply to any given group multiplication table—if the output is "yes", then we know the group is free abelian. The existence of such a property depends on the cardinal properties of κ; generally, however, we are out of luck: there is no such property, and hence the only way to determine definably whether a group is free abelian is by asking the question "does the group have a basis?".

Theorem 3.2.5 ([56, Thm 1.4 (1)] ($V{=}L$)).

1. *If κ is a successor cardinal or the least inaccessible cardinal, then $\mathcal{G}(\kappa)$ is Σ^1_1-complete.*
2. *If κ is not weakly compact, then $\mathcal{G}(\kappa)$ is $\utilde{\boldsymbol{\Sigma}}^1_1$-complete*[12].

Part (1) of Theorem 3.2.5 yields the following corollary, which is interesting in its own right: it motivates our theorems in chapter 4.

Corollary 3.2.6 ([56, Cor 1.5] ($V{=}L$))**.** *If κ is a successor cardinal and $X \in \Delta^1_1(2^\kappa)$, then there is a group $G \in \mathcal{G}(\kappa)$ with no X-computable basis.*

Proof. Assume not. Take a computable listing of all pairs of $\Sigma^0_1(L_\kappa; X)$ formulas $(\varphi_\alpha, \theta_\alpha)$ of one free variable, and denote the sets they define by S_α and P_α. The listing (S_α, P_α) contains all X-computable sets (cf. Theorem 3.1.25 and Definition 3.1.26), and the relation "$S_\alpha = L_\kappa \setminus P_\alpha$" is arithmetical in X via "$(\forall x)(\varphi_\alpha(x) \iff \neg\theta_\alpha(x))$". But now we can give a $\Pi^1_1(L_\kappa)$-characterisation of the class of free abelian groups: since X is Δ^1_1, every S_α and P_α can be expressed as a Π^1_1-set, and so

$$G \in \mathcal{G}(\kappa) \iff (\exists \alpha < \kappa)(S_\alpha = L_\kappa \setminus P_\alpha \wedge \psi^*(G, P_\alpha))$$

is a Π^1_1-relation[13]. This is impossible since $\mathcal{G}(\kappa)$ is $\Sigma^1_1(L_\kappa)$-complete. □

In the following chapter, we make this precise by exhibiting uncountable free abelian groups whose bases defy computation by a given Δ^1_1-oracle. Specifically, given a regular cardinal κ and a Δ^1_1-oracle X, we construct an uncountable free abelian group of universe κ which is κ-computable, while none of its bases can be computed from X.

[12]Boldface completeness admits a fixed subset parameter; cf. [56, p. 4487]: there exists $D \subset \kappa$ for which $\mathcal{G}(\kappa)$ is $\Sigma^1_1(D)$-complete; hence every $\Sigma^1_1(D)$-set is reducible to $\mathcal{G}(\kappa)$ via a D-computable function.

[13]It is even arithmetical in X.

Chapter 4

Constructing Complicated Free Abelian Groups

In this chapter we answer an open question from [56]: we extend results found by Greenberg, Turetsky, and Westrick in [56] and investigate effective properties of bases of uncountable free abelian groups. Assuming $V{=}L$, we show that if κ is a regular uncountable cardinal and X is a $\Delta^1_1(L_\kappa)$ subset of κ, then there is a κ-computable free abelian group whose bases cannot be computed by X. Unlike in [56], we give a direct construction.

The results in this chapter were obtained in joint work with Noam Greenberg, Saharon Shelah, and Daniel Turetsky, and we thank them for allowing us to include them here.

4.1 Constructing Bases Recursively

The recursive construction of a basis of a vector space is taught to first-year students of linear algebra: at each step, add some vector which does not lie in the span of the previous ones chosen. This "algorithm" works equally well in finite, countable and uncountable-dimensional vector spaces. In the uncountable case, this can be made precise using the framework introduced by Greenberg and Knight for uncountable computable structure theory, using admissible computability on uncountable cardinals [54]. In fact, just like the countable case, a single Turing jump (relative to the diagram of the vector space) suffices. In particular, for every cardinal κ, every κ-computable vector space (over a κ-computable field) has a basis which is definable in L_κ.

What happens when we consider free abelian groups, objects which appear just as simple as vector spaces? In the countable case, based on a strong notion of independence introduced by Pontryagin [130], a similar algorithm can be performed; Downey and Melnikov [26] used this notion to show that a single Turing jump suffices to build a basis of a countable free abelian group. This algorithm, however, fails badly in the uncountable case, because difficulties can arise in limit steps: it is possible to generate an ω-sequence $b_0, b_1, \dots$ of elements of a free abelian group such that every finite initial segment of the sequence is extendible to a basis of the group, but the whole sequence cannot. That is, the construction, which appears perfectly fine at every finite step, "explodes" at the limit step.

In [56], the authors showed that this difficulty is fundamental: in general, there is no way to construct bases of free abelian groups in a step-by-step recursive, or definable, construction. Formally, what they showed (under the standard assumption that $V{=}L$) is that for most uncountable regular cardinals κ, there are κ-computable free abelian groups which have no bases definable over L_κ; in fact, no single κ-Turing degree which is $\Delta^1_1(L_\kappa)$ can serve as an oracle which computes bases for all κ-computable free abelian groups. This result has two shortcomings:

(1) It does not apply to all regular uncountable cardinals κ; and

(2) it does not rely on a direct construction.

These two are related. The argument in [56] relies on the complexity of *identifying* which groups are free. The authors show that if κ is not weakly compact, then the collection of free abelian groups is $\underset{\sim}{\mathbf{\Sigma}}^1_1(L_\kappa)$-complete, and in many cases (such as successor cardinals, or the least inaccessible), is (lightface) $\Sigma^1_1(L_\kappa)$-complete. In these cases, this completeness result implies that searching for bases cannot be restricted to the sets computable from a fixed $\Delta^1_1(L_\kappa)$ oracle. It is not clear from the construction, though, how to directly build, given a $\Delta^1_1(L_\kappa)$ oracle X, a κ-computable free abelian group G with no X-computable basis. And this approach cannot work for weakly compact cardinals κ; the very compactness property of these cardinals implies that it is in fact not very difficult to identify which groups are free; in fact, a single jump suffices. This leaves the original question, of the complexity of bases, open for these cardinals. Here we fully answer the original question:

Theorem 4.1.1 ($V{=}L$)**.** *Let κ be regular and uncountable. For any $X \in 2^\kappa$ which is $\Delta^1_1(L_\kappa)$ there is a κ-computable free abelian group, none of whose bases is X-computable.*

The proof of the theorem is done by a direct construction.

4.2 Preliminaries, and the Plan of the Proof

The proof of Theorem 4.1.1 is the combination of two distinct parts: we will first reduce the problem to one in "computable set theory", one which abstracts away the algebraic part; and then give a construction solving the reduced problem. Throughout this chapter, we assume $V{=}L$.

We start by recalling some basic concepts that were used in [56]. When considering free abelian groups, we will omit the adjective "abelian" and just call these groups free.

Definition 4.2.1. *Let G be a group. A subgroup H of G* **detaches** *in G if there is a G-subgroup K such that $G = H \oplus K$. We write $H \mid G$.*

Any subgroup of a free abelian group is free. A subgroup H of a free abelian group G detaches in G if and only if some (equivalently any) basis of H can be extended to a basis of G, if and only if the quotient G/H is free. If H detaches in G then H detaches in any subgroup K of G containing H.

Definition 4.2.2. *A sequence $\langle G_\alpha \rangle_{\alpha<\gamma}$ of groups of some ordinal length γ is* **increasing** *if $\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$; it is* **continuous** *if for all limit $\alpha < \gamma$, $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. A* **filtration** *of a group G is a sequence $\overline{G} = \langle G_\alpha \rangle$ such that $\overline{G}$ is increasing, continuous, $G = \bigcup_\alpha G_\alpha$, and $|G_\alpha| \leq |\alpha|$ for all α.*

If γ is regular and G is a group of universe γ then all filtrations of G agree on a club; in fact, by Lemma 2.2.5, $G_\alpha = G \cap \alpha$ for club many α.

Definition 4.2.3. *Let $\overline{G} = \langle G_\alpha \rangle_{\alpha<\gamma}$ be increasing and continuous. The* **detachment set** *of $\overline{G}$ is*

$$\mathrm{Div}(\overline{G}) = \{\alpha < \gamma \,:\, \forall \beta \in (\alpha, \gamma)\ (G_\alpha \mid G_\beta)\}\,.$$

If γ is regular and $\overline{G}, \overline{G}'$ are two filtrations of a group of universe γ, then $\mathrm{Div}(\overline{G})$ and $\mathrm{Div}(\overline{G}')$ agree on a club.

The following can essentially be found in [30]; see [31, IV.1.7].

Proposition 4.2.4. *Suppose γ is a limit ordinal and let $\overline{G} = \langle G_\alpha \rangle_{\alpha<\gamma}$ be a filtration of a group G_γ. Suppose G_α is free for all $\alpha < \gamma$.*

(1) If $\mathrm{Div}(\overline{G})$ contains a γ-club, then G_γ is free.

(2) If γ is regular and G_γ is free, then $\mathrm{Div}(\overline{G})$ contains a γ-club.

Remark. Let γ be a limit ordinal; let $\overline{G} = \langle G_\alpha \rangle_{\alpha<\gamma}$ be a filtration of a group G_γ. Suppose that $\mathrm{Div}(\overline{G})$ contains a club of γ. Then

$$\mathrm{Div}(\overline{G}) = \{\alpha < \gamma \,:\, G_\alpha \mid G_\gamma\}\,.$$

The proof of Proposition 4.2.4 is effective. This gives the following:

Proposition 4.2.5. *Let κ be regular and uncountable; Let $\overline{G} = \langle G_\alpha \rangle_{\alpha<\kappa}$ be a filtration of a free group G (with universe κ). The bases of G and the club subsets of $\mathrm{Div}(\overline{G})$ are equicomputable modulo $\overline{G}$. That is:*

- *If C is club in $\mathrm{Div}(\overline{G})$ then there is a basis B of G s.t. $B \leq_\kappa C \oplus \overline{G}$;*
- *If B is a basis of G then there is a club C of $\mathrm{Div}(\overline{G})$ s.t. $C \leq_\kappa B \oplus \bar{G}$.*

The reductions in both directions are uniform. Briefly, given a club subset $C = \{\alpha_i : i < \kappa\}$ of $\mathrm{Div}(\overline{G})$, we can present G as the direct sum of κ-finite groups H_i with $G_{\alpha_i} = \bigoplus_{j<i} H_j$, and combine bases of the groups H_i to a basis of G. In the other direction, given a basis B of G, we let C be the collection of α such that $B \cap \alpha$ generates G_α.

From now, we fix a regular uncountable cardinal κ. Recall the lexicographic ordering on 2^κ: $S < T$ if for the greatest $\delta < \kappa$ such that $S \restriction \delta = T \restriction \delta$ we have $\delta \in T \setminus S$. If $\langle S_i \rangle_{i \leq i^*}$ is a lexicographically nondecreasing sequence (for some ordinal i^*), $i < j \leq i^*$ and $S_i \restriction \delta = S_j \restriction \delta$ for some $\delta < \kappa$, then $S_r \restriction \delta$ is constant for all $r \in [i, j]$.

The lexicographic ordering is complete. A lexicographically nondecreasing sequence $\langle S_i \rangle_{i \le i^*}$ is **continuous** if for all limit $j \le i^*$, S_j is the least upper bound of $\langle S_i \rangle_{i<j}$. This means that if $j \le i^*$ is a limit, $\delta < \kappa$ and $S_i \restriction \delta$ is constant for some final segment of $i < j$, then $S_i \restriction \delta + 1$ is also constant for some final segment of $i < j$.

A **nondecreasing approximation** of a set $S \subseteq \kappa$ is a uniformly κ-computable sequence $\langle S_i \rangle_{i<\kappa}$ which is lexicographically nondecreasing such that $S = \sup_{i<\kappa} S_i$. Sets which have nondecreasing approximations are called **κ-left-c.e.** If S is κ-left-c.e. then it has a nondecreasing approximation which is also continuous.

We say that a set $S \subseteq \kappa$ is **nowhere stationary** if for all $\alpha \le \kappa$ of uncountable cofinality, $S \cap \alpha$ is nonstationary in α.

To avoid repetition, we make the following definition.

Definition 4.2.6. *We say that a nowhere stationary, κ-left-c.e. set $S \subseteq \kappa$ is* **nicely thin** *if there is a continuous nondecreasing approximation $\langle S_i \rangle_{i<\kappa}$ of S such that for all $i < \kappa$, S_i is nowhere stationary, and every $\alpha \in S_i$ is a limit ordinal of countable cofinality.*

As promised, the following proposition distills the algebraic aspects of our construction.

Proposition 4.2.7. *If $S \subset \kappa$ is nicely thin, then there is a κ-computable free abelian group G and a $\Delta^0_2(L_\kappa)$ filtration $\overline{G}$ of G such that $\mathrm{Div}(\overline{G}) = \kappa \setminus S$.*

Proposition 4.2.8. *Let X be $\Delta^1_1(L_\kappa)$. There is a nicely thin set $S \subseteq \kappa$ which intersects every X-computable club set.*

Proof of Theorem 4.1.1. Let X be $\Delta^1_1(L_\kappa)$; we may assume that $X \ge_\kappa \emptyset'$. Let $S \subseteq \kappa$ be a nicely thin set given by Proposition 4.2.8. Let G and $\overline{G}$ be given by Proposition 4.2.7 from this set S. Since $\overline{G}$ is $\Delta^0_2(L_\kappa)$ and $X \geq_\kappa \emptyset'$, we have $X \geq_\kappa \overline{G}$. Suppose that B is a basis of G. By Proposition 4.2.5, there is a club set $C \subseteq \mathrm{Div}(\overline{G})$ such that $C \leq_\kappa B \oplus \overline{G}$. So C is disjoint from $S = \kappa \setminus \mathrm{Div}(\overline{G})$, whence $C \not\leq_\kappa X$. Hence $B \not\leq_\kappa X$ as well. □

The next two sections are devoted to the proofs of Propositions 4.2.7 and 4.2.8, respectively.

4.3 Building Groups From Left-c.e. Sets

In this section we prove Proposition 4.2.7. The main tool we use is the "twisting" of an increasing ω-sequence of free groups.

Proposition 4.3.1 ([56, Prop 2.16])**.** *There is a κ-computable process which given a κ-finite increasing ω-sequence $\langle H_n \rangle$ of free groups such that $H_n \mid H_{n+1}$ for all n, produces a κ-finite free group $G \supseteq H_\omega = \bigcup_n H_n$ (with $|G| = |H_\omega|$) such that every H_n detaches in G but H_ω does not detach in G.*

This construction relies on a copy G of $\mathbb{Z}^{(\omega+1)}$ such that $G/\mathbb{Z}^{(\omega)}$ is not free; Pontryagin's criterion implies that each finitely generated subgroup of $\mathbb{Z}^{(\omega)}$ detaches in G. In Proposition 4.3.1, note that each H_n detaches in H_ω.

First, we show how to prove Proposition 4.2.7 in the simpler case when S is κ-computable (rather than merely κ-left-c.e.). We can then produce both G and $\overline{G}$ to be κ-computable. The construction follows the proof of [56, Theorem 3.1]. We define the sequence $\langle G_\alpha \rangle_{\alpha<\kappa}$ by effective κ-recursion (Σ_1 recursion over L_κ). During the construction we ensure that each G_α is free, and for all $\beta < \alpha$, if $\beta \notin S$ then $G_\beta \mid G_\alpha$. The construction has three cases.

1. α is a limit: We must define $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Since $S \cap \alpha$ is nonstationary in α and $\mathrm{Div}(\langle G_\beta \rangle_{\beta<\alpha}) \supseteq \alpha \setminus S$, G_α is free (Proposition 4.2.4). Further, if $\beta \in \alpha \setminus S$ then as $\beta \in \mathrm{Div}(\langle G_\beta \rangle_{\beta<\alpha})$, we have $G_\beta \mid G_\alpha$ (section 4.2).

2. Defining $G_{\alpha+1}$ when $\alpha \notin S$: We let $G_{\alpha+1} = G_\alpha \oplus \mathbb{Z}$. Since G_α is free, so is $G_{\alpha+1}$. We certainly have $G_\alpha \mid G_{\alpha+1}$, and for all $\beta \in \alpha \setminus S$, by induction, $G_\beta \mid G_\alpha$, and so $G_\beta \mid G_{\alpha+1}$ as well.

3. Defining $G_{\alpha+1}$ when $\alpha \in S$: Since $\alpha \in S$, $\mathrm{cf}(\alpha) = \omega$, so we can effectively find a cofinal ω-sequence $\langle \beta_n \rangle$ in α; by replacing each β_n by its successor, we may assume that each $\beta_n \notin S$. Thus, by induction, for all n, $G_{\beta_n} \mid G_{\beta_{n+1}}$, and $G_\alpha = \bigcup_n G_{\beta_n}$. We then apply Proposition 4.3.1 to the sequence $\langle G_{\beta_n} \rangle$ to obtain a free group $G_{\alpha+1}$ extending G_α such that $G_\alpha \nmid G_{\alpha+1}$, yet $G_{\beta_n} \mid G_{\alpha+1}$ for all n. If $\beta \leq \alpha$ and $\beta \notin S$, then $\beta < \alpha$; for sufficiently large n we have $\beta_n > \beta$. By induction, $G_\beta \mid G_{\beta_n}$, and by construction, $G_{\beta_n} \mid G_{\alpha+1}$, so $G_\beta \mid G_{\alpha+1}$ as required.

We define $G = G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$. It is κ-computable since it is the union of a κ-computable sequence of κ-finite groups. Case 1 above holds for κ as well: $\mathrm{Div}(\langle G_\alpha \rangle) \supseteq \kappa \setminus S$; since S is nonstationary in κ, G is free. On the other hand, for all $\alpha \in S$, by construction, $G_\alpha \nmid G_{\alpha+1}$, so $\beta \notin \mathrm{Div}(\langle G_\alpha \rangle)$; overall, we see that $\mathrm{Div}(\langle G_\alpha \rangle) = \kappa \setminus S$ as required.

We now consider the general case, when S is κ-left-c.e. The construction is an elaboration on the proof of [56, Theorem 3.2]. The idea is that if at stage $s < \kappa$ we see a change in S at some small $\alpha < s$, then we keep the group that we have constructed so far, but "squash" the filtration so that the old G_s becomes a subgroup of the new G_α. The fact that the smallest such change is into S allows us to introduce a twist (otherwise we would need to remove a twist, which is impossible).

Let $S \subset \kappa$ be nicely thin, and let $\langle S_t \rangle_{t<\kappa}$ be an approximation satisfying Definition 4.2.6. For brevity, we let $S_\kappa = S$.

We now construct, by recursion on $t \leq \kappa$, filtrations $\overline{G^t} = \langle G^t_\alpha \rangle_{\alpha \leq \gamma(t)}$ (for ordinals $\gamma(t) \leq t$), satisfying:

(a) $G^s_{\gamma(s)} \subseteq G^t_{\gamma(t)}$ for $s \leq t \leq \kappa$, and if t is a limit then $G^t_{\gamma(t)} = \bigcup_{s<t} G^s_{\gamma(s)}$. That is, the sequence $\langle G^s_{\gamma(s)} \rangle_{s \leq \kappa}$ is increasing and continuous.

(b) $G^t_{\gamma(t)}$ is free and $\mathrm{Div}(\overline{G^t}) = \gamma(t) \setminus S_t$ for all $t \leq \kappa$. Indeed, for all $\beta < \gamma(t)$, if $\beta \notin S_t$ then $G^t_\beta \mid G^t_{\gamma(t)}$, while if $\beta \in S_t$ then $G^t_\beta \nmid G^t_{\beta+1}$.

(c) For all $s \leq t \leq \kappa$ and $\alpha \leq \gamma(s)$, if $S_s \restriction \alpha = S_t \restriction \alpha$ then $\alpha \leq \gamma(t)$ and $G^s_\alpha = G^t_\alpha$.

(d) For all $s < t \leq \kappa$ and $\alpha \leq \gamma(t)$, if $S_s \restriction \alpha \neq S_t \restriction \alpha$ then $G^s_{\gamma(s)} \subseteq G^t_\alpha$.

We first consider the successor case. Suppose that $\overline{G^s}$ has been defined for all $s \leq t$; we show how to define $\overline{G^{t+1}}$. First, we let

$$\delta = \max\{\beta \leq \gamma(t) \mid S_t \restriction \beta = S_{t+1} \restriction \beta\}.$$

For $\beta \leq \delta$ we let $G^{t+1}_\beta = G^t_\beta$. We let $\gamma(t+1) = \delta + 1$, and so we need to define $G^{t+1}_{\delta+1}$. There are two sub-cases. If $\delta \notin S_{t+1}$, then $\delta \notin S_t$, and so the maximality of δ shows that $\delta = \gamma(t)$. In this sub-case we simply let $G^{t+1}_{\delta+1} = G^t_\delta \oplus \mathbb{Z}$.

Suppose that $\delta \in S_{t+1}$. Either $\delta = \gamma(t)$, or $\delta \notin S_t$ (by the maximality of δ). In either case, $G^t_\delta \mid G^t_{\gamma(t)}$. Write $G^t_{\gamma(t)} = G^t_\delta \oplus K$ (where K may be trivial). Since $\mathrm{cf}(\delta) = \omega$ (as $\delta \in S_{t+1}$), we choose a sequence $\langle \beta_n \rangle$ cofinal in δ and disjoint from S_{t+1}. By (b), $G^t_{\beta_n} \mid G^t_{\beta_{n+1}}$ for all $n < \omega$, so we appeal to Proposition 4.3.1 to get a free group $G \supset G^t_\delta$ in which every G_{β_n} detaches, but $G^t_\delta \nmid G$. We let $G^{t+1}_{\delta+1} = G \oplus K$.

By design, $G^t_{\gamma(t)} \subseteq G^{t+1}_{\delta+1} = G^{t+1}_{\gamma(t+1)}$, verifying (a) for stage $t+1$. We verify (b). For $\beta < \delta$, if $\beta \in S_{t+1}$ then $\beta \in S_t$ and then $G^t_\beta \nmid G^t_{\beta+1}$, which shows that $\beta \notin \operatorname{Div}(\overline{G^{t+1}})$; if $\beta \notin S_{t+1}$ then $\beta \notin S_t$. If $\delta \notin S_{t+1}$ then $G^t_\beta \mid G^t_\delta \mid G^{t+1}_{\delta+1}$, showing that $\beta \in \operatorname{Div}(\overline{G^{t+1}})$. If $\delta \in S_{t+1}$ then for some n, $\beta < \beta_n$, and $G^t_\beta \mid G^t_{\beta_n} \mid G^{t+1}_{\delta+1}$, showing that $\beta \in \operatorname{Div}(\overline{G^{t+1}})$. (c) between t and $t+1$ follows from the definition of δ. In general, for $s < t$, if $\alpha \leq \gamma(s)$ and $S_s \restriction \alpha = S_{t+1} \restriction \alpha$, then $S_s \restriction \alpha = S_t \restriction \alpha$ as well; by induction, $\alpha \leq \gamma(t)$, and so $\alpha \leq \delta$ and $G^s_\alpha = G^t_\alpha = G^{t+1}_\alpha$. For (d), let $s < t+1$, $\alpha \leq \gamma(t+1)$, and suppose that $S_s \restriction \alpha \neq S_{t+1} \restriction \alpha$; we need to show that $G^s_{\gamma(s)} \subseteq G^{t+1}_\alpha$. In case $\alpha = \gamma(t+1)$, this follows from (a). Otherwise, $\alpha \leq \delta$, and so $\alpha \leq \gamma(t)$, and $G^{t+1}_\alpha = G^t_\alpha$. By the definition of δ, it must be that $S_s \restriction \alpha \neq S_t \restriction \alpha$. Then by induction, $G^s_{\gamma(s)} \subseteq G^t_\alpha$.

Now suppose that $t \leq \kappa$ is a limit ordinal, and suppose that the filtrations $\overline{G^s}$ have been defined for all $s < t$, and satisfy the conditions described. We now describe how to define $\overline{G^t}$. Let

$$\Delta = \{\beta \mid (\exists s < t)\,(\beta \leq \gamma(s) \wedge S_s \restriction \beta = S_t \restriction \beta)\}\,.$$

If $s < t$ witnesses that $\beta \in \Delta$, then s witnesses that every $\epsilon < \beta$ is in Δ as well, and so Δ is actually an ordinal. We let $\gamma(t) = \Delta$.

We argue that Δ is in fact a limit ordinal. This relies on the approximation $\langle S_s \rangle$ being continuous. Suppose, for a contradiction, that $\delta = \max \Delta = \Delta - 1$ exists. Let $s < t$ witness that $\delta \in \Delta$. Then $S_r \restriction \delta$ is constant for $r \in [s, t]$. By definition of Δ we have $\delta \leq \gamma(s)$. Since (c) holds by induction, for all $r \in [s, t)$ we have $\delta \leq \gamma(r)$, that is, every $r \geq s$ also witnesses that $\delta \in \Delta$. Since the approximation is continuous, for sufficiently late r, we have $S_r(\delta) = S_t(\delta)$, that is, $S_r \restriction \delta + 1$ is constant on a final segment of $r < t$. So we may assume that $S_s \restriction \delta + 1 = S_t \restriction \delta + 1$. Now by our instructions for the successor case, we have $\gamma(s + 1) \geq \delta + 1$, so stage $s + 1$ witnesses that $\delta + 1 \in \Delta$, a contradiction.

Now for all $\beta < \gamma(t) = \Delta$, we define $G^t_\beta = G^s_\beta$ for any $s < t$ witnessing that $\beta \in \Delta$; again, as $S_r \restriction \beta = S_t \restriction \beta$ for all $r \in [s, t)$, by (c) G^r_β is constant for all such r, so G^t_β is well-defined, and $\langle G^t_\beta \rangle_{\beta<\gamma(t)}$ is increasing and continuous; we let $G^t_{\gamma(t)} = \bigcup_{\beta<\gamma(t)} G^t_\beta$, and this defines $\overline{G^t}$, which is indeed a filtration. We verify that the conditions above hold for t.

(c) is by construction. For (d), let $s < t$ and $\alpha \leq \gamma(t)$, and suppose that $S_s \restriction \alpha \neq S_t \restriction \alpha$. As above, we now assume that $\alpha < \gamma(t)$, as the case $\alpha = \gamma(t)$ will follow from (a), which we will soon verify. Since $\alpha \in \Delta$, find some $r < t$ such that $S_r \restriction \alpha = S_t \restriction \alpha$ and $\alpha \leq \gamma(r)$. Then $s < r < t$. By induction, $G^s_{\gamma(s)} \subseteq G^r_\alpha$, and $G^r_\alpha = G^t_\alpha$.

For (a), Let $s < t$. Since $\gamma(t) \notin \Delta$, and $\gamma(t)$ is a limit, there is some $\beta < \gamma(t)$ such that $S_s \restriction \beta \neq S_t \restriction \beta$. Again let $r \in (s, t)$ such that $S_r \restriction \beta = S_t \restriction \beta$ and $\beta \leq \gamma(r)$. By induction,

$$G^s_{\gamma(s)} \subseteq G^r_\beta = G^t_\beta \subseteq G^t_{\gamma(t)}.$$

Finally, we verify (b). Let $\beta < \gamma(t)$. Suppose that $\beta \in S_t$. Take $s < t$ witnessing that $\beta + 1 \in \Delta$. Then $\beta \in S_s$ and by induction, $G^s_\beta \nmid G^s_{\beta+1}$; and $G^t_\beta = G^s_\beta$ and $G^t_{\beta+1} = G^s_{\beta+1}$. Suppose that $\beta \notin S_t$, and let $\delta \in (\beta, \gamma(t))$. Let $s < t$ witness that $\delta \in \Delta$. Then $\delta \notin S_s$, and so $G^s_\beta \mid G^s_\delta$; so $G^t_\beta \mid G^t_\delta$, showing that $\beta \in \mathrm{Div}(\langle G^t_\alpha \rangle_{\alpha<\gamma(t)})$. Now by assumption, the set S_t is nonstationary in $\gamma(t)$, and so $G^t_{\gamma(t)}$ is free, and $\mathrm{Div}(\overline{G^t}) = \gamma(t) \setminus S_t$.

This completes the construction. There is one thing left to show: that $\gamma(\kappa) = \kappa$. This fundamentally follows from the regularity of κ, which implies that the approximation $\langle S_t \rangle$ of $S = S_\kappa$ is **tame**: for all $\beta < \kappa$ there is some $t < \kappa$ such that $S_t \restriction \beta = S_\kappa \restriction \beta$. [This is proved by induction on β; if this is known for all $\alpha < \beta$, then the function taking such α to the least t for which $S_t \restriction \alpha = S_\kappa \restriction \alpha$ must be bounded below κ.] So now we show by induction on $\beta < \kappa$ that $\beta \leq \gamma(s)$ for some s for which $S_s \restriction \beta = S_\kappa \restriction \beta$ (which by (c) for s and κ implies that $\beta \leq \gamma(\kappa)$). If this is known for β, then by taking a sufficiently late s, we may assume that $S_s \restriction \beta + 1 = S_\kappa \restriction \beta + 1$ as well; then by construction, $\beta + 1 \leq \gamma(s+1)$. Now suppose that β is a limit ordinal. For all $\alpha < \beta$, let s_α be the least s for which $\alpha \leq \gamma(s)$ and $S_s \restriction \alpha = S_\kappa \restriction \alpha$. Then the sequence $\langle s_\alpha \rangle_{\alpha < \beta}$ is nondecreasing; let $s^* = \sup_{\alpha<\beta} s_\alpha$. Then $S_{s^*} \restriction \beta = S_\kappa \restriction \beta$, and by (c), for all $\alpha < \beta$, $\alpha \leq \gamma(s^*)$, so $\beta \leq \gamma(s^*)$.

The restriction of the construction to $t < \kappa$ is κ-computable. In particular, the sequence $\langle G^s_{\gamma(s)} \rangle$ is κ-computable, and so its union, which is G^κ_κ, is κ-computable; by (b) at $t = \kappa$, it is free. The filtration $\langle G^\kappa_\alpha \rangle_{\alpha<\kappa}$ is $\Delta^0_2(L_\kappa)$ and by (b), $\mathrm{Div}(\overline{G^\kappa}) = \kappa \setminus S_\kappa$ as required. This completes the proof of Proposition 4.2.7.

Note. There is no reason to believe that $\mathrm{Div}(\langle G^s_{\gamma(s)} \rangle) = \kappa \setminus S$; they agree on a club, but that club may fail to be κ-computable, it is merely $\Delta^0_2(L_\kappa)$.

4.4 Constructing Fat Thin Sets

In this section we prove Proposition 4.2.8. As in the previous section, we first consider the construction under some simplifying assumptions. First, we review the basic tools of fine-structure theory that were used in [56] and will use again below. They are taken from Jensen's original paper [64].

Definition 4.4.1. *The class E consists of all the singular ordinals α such that for some $\beta > \alpha$:*

- $L_\beta \models \mathsf{ZF}^-$*;*
- *α is the greatest cardinal of L_β;*
- *there exists $p \in L_\beta$ such that L_β is the least fully elementary substructure $M \prec L_\beta$ with $p \in M$ and $M \cap \alpha$ transitive.*

Each $\alpha \in E$ has countable cofinality; in fact, if $\alpha \in E$, witnessed by β, then there is a cofinal ω-sequence in α definable over $L_{\beta+1}$. Thus, once we see that an ordinal α is singular, we can effectively tell if $\alpha \in E$ or not. Thus, for any regular κ, $E \cap \kappa$ is κ-c.e.; if κ is a successor cardinal, then $E \cap \kappa$ is κ-computable. The following lemma is used to produce elements of E:

Lemma 4.4.2. *Let κ be regular and uncountable; let $q \in L_{\kappa^+}$. Let M be the least elementary substructure of L_{κ^+} such that $q \in M$ and $M \cap \kappa$ is transitive. Let $\pi\colon M \to L_\beta$ be the Mostowski collapse; let $\alpha = \pi(\kappa) = M \cap \kappa$. Then $\alpha \in E$, witnessed by β.*

The deep fact about E that is used throughout is [64, Theorem 5.1]:

Theorem 4.4.3 (Jensen)**.** *The class E does not reflect at any singular ordinal. That is, if α is singular then $E \cap \alpha$ is nonstationary in α.*

For a proof, see Corollary 2.2.8. On the other hand, E is stationary in every regular cardinal.

Toward the full proof of Proposition 4.2.8, we give the proof when κ is a successor cardinal. In that case, $E \cap \kappa$ is κ-computable, and so we can make the desired set S κ-computable as well. To simplify even further, rather than meeting X-computable clubs for some possibly quite complicated X, we consider a simpler collection of sets, namely the first-order definable ones:

Proposition 4.4.4. *Let κ be a successor cardinal. There is a κ-computable set S which is nowhere stationary but intersects every club of κ which is first-order definable over L_κ.*

This implies that there is a κ-computable free group with no first-order definable basis.

Proof. The set S is constructed by recursion. At stage $\delta < \kappa$, we will have already defined $S \restriction \delta$. If $\delta \notin E$ then $\delta \notin S$. Suppose that $\delta \in E$. Then we let $\delta \in S$ if and only if there is some $C \subseteq \delta$, closed and unbounded in δ, which is first-order definable over L_δ, which is disjoint from $S \cap \delta$.

Because E is κ-computable, the construction is κ-computable, and so S is κ-computable.

Let us show that S intersects every club of κ, first-order definable over L_κ. Let C be such a club. Let M be the smallest elementary substructure of L_{κ^+} which contains the parameters used for the definition of C, and such that $M \cap \kappa$ is transitive. Let $\pi\colon M \to L_\beta$ be the Mostowski collapse, and let $\alpha = \pi(\kappa) = M \cap \kappa$. By Lemma 4.4.2, $\alpha \in E$. Now $\pi(C) = C \cap \alpha$ is a club of α which is first-order definable over $L_\alpha = \pi(L_\kappa)$. Also note that $\alpha \in C$ (as C is closed). If $\pi(C) \cap (S \cap \alpha) \neq \emptyset$ then some $\gamma < \alpha$ is an element of $S \cap C$, in which case we are done. Otherwise, by construction, we put α into S, so in this case $\alpha \in C \cap S$.

It remains to show that S is nowhere stationary. By construction, $S \subseteq E$, and E does not reflect at any singular ordinal (Theorem 4.4.3), so it suffices to show that $S \cap \lambda$ is nonstationary in λ, for every regular cardinal $\lambda \leq \kappa$.

By constructing an increasing, continuous sequence of elementary submodels of L_{κ^+}, we obtain a closed set $D \subseteq \kappa$ such that $D \cap \lambda$ is unbounded in λ for every regular $\lambda \leq \kappa$, and such that for all $\alpha \in D$ there is a model $M_\alpha \prec L_{\kappa^+}$ with $\alpha = M_\alpha \cap \kappa$. It suffices to show that $D \cap S = \emptyset$. Let $\alpha \in D$; let $\pi \colon M_\alpha \to L_\beta$ be the Mostowski collapse. Let $C \subseteq \alpha$ be a club of α, first-order definable over L_α. Then $C \in L_\beta$, and $\pi^{-1}(C)$ is a club of κ, first-order definable over L_κ. We have just proved that $S \cap \pi^{-1}(C) \neq \emptyset$. Note that $S \cap \alpha \in M_\alpha$ (as it is definable over L_κ). By elementarity of M_α, there is some $\gamma < \alpha$ in $\pi^{-1}(C) \cap S$. Then $\gamma \in C \cap (S \cap \alpha)$. Thus, $S \cap \alpha$ meets all clubs of α which are first-order definable over L_α, and so even if $\alpha \in E$, the construction would instruct us to keep α out of S. $\square$

Now, to prove Proposition 4.2.8, we need to overcome two obstacles:

- If κ is not a successor cardinal, we need to deal with the fact that $E \cap \kappa$ is not κ-computable, but merely κ-c.e.
- We need to diagonalise against all X-computable clubs for some X which is $\Delta^1_1(L_\kappa)$, not just first-order definable ones.

For the first difficulty, we fix a κ-computable enumeration $\langle E_t \rangle_{t<\kappa}$ of E, and repeat the construction above at each stage $t \leq \kappa$, giving us a set S_t for each such t; we show that this is an approximation as required. For the second, we use a technique of approximating (or reflecting) Δ^1_1 sets that was used by Johnston [65, Thm 4.43] and S. Friedman and his co-authors [40, Lem 2.5].

Fix a $\Delta^1_1(L_\kappa)$ set X. Thus, there are two first-order formulas φ and ψ, with parameters in L_κ, such that $(\exists Y)\psi(-, Y)$ and $(\exists Y)\varphi(-, Y)$ define X and its complement respectively, where the variable Y ranges over subsets of κ, and for each $Y \subseteq \kappa$, the formula $\varphi(-, Y)$ is evaluated in the structure $(L_\kappa; \in, Y)$ (with Y interpreting a unary predicate).

Fix ordinals $\alpha < \beta$ such that $L_\beta \models \mathsf{ZF}^-$ and α is regular in L_β and is the largest cardinal of L_β. Also assume that the parameters used for the formulas φ and ψ are elements of L_α. We let:

$$A_\alpha^\beta = \{\gamma < \alpha \,|\, (\exists y \in L_\beta)(y \subseteq \alpha \wedge L_\alpha \models \varphi(\gamma, y))\}$$

$$B_\alpha^\beta = \{\gamma < \alpha \,|\, (\exists y \in L_\beta)(y \subseteq \alpha \wedge L_\alpha \models \psi(\gamma, y))\}$$

If $\alpha \in E$ then we write A_α and B_α for A_α^β and B_α^β for the unique β which witnesses that $\alpha \in E$. We say that $\alpha \in E$ is **good** if $B_\alpha = \alpha \setminus A_\alpha$. In this case, A_α is our guess for X at level α. The guess is correct on a club since:

Lemma 4.4.5. *Suppose that M is an elementary substructure of L_{κ^+} (containing the parameters for φ and ψ), and $M \cap \kappa \in \kappa$. Let $\alpha = M \cap \kappa$, and let L_β be the Mostowski collapse of M. Then $B_\alpha^\beta = \alpha \setminus A_\alpha^\beta$, and $A_\alpha^\beta = X \cap \alpha$.*

We now provide the proof of Proposition 4.2.8. Let κ be regular; fix X, φ and ψ as above. Let $\langle E_t \rangle$ be a κ-computable enumeration of $E \cap \kappa$: for all $t \leq \kappa$, E_t is the collection of $\alpha < t$ which are witnessed to be in E by some $\beta < t$. The facts about this enumeration that we use are:

(1) If $s \leq t \leq \kappa$ then $E_s \subseteq E_t$; if $t \leq \kappa$ is limit then $E_t = \bigcup_{s<t} E_s$; $E_\kappa = E \cap \kappa$.
(2) For any cardinal $\lambda < \kappa$, for all $t \geq \lambda$, $E_t \cap \lambda = E \cap \lambda$.

Now for each $t \leq \kappa$ we define a set $S_t \subseteq E_t$ by recursion. For $\alpha < t$, if $S_t \restriction \alpha$ has already been defined, then we set $\alpha \in S_t$ if and only if $\alpha \in E_t$, α is good, and there is a club of α, disjoint from $S_t \cap \alpha$, which is Δ_1^0 definable in the structure $(L_\alpha; \in, A_\alpha)$.

We first observe that S_κ intersects every X-computable club. Let C be an X-computable club. As above, let M be a minimal elementary submodel of L_{κ^+} containing both the parameters for the definitions of X and its complement, and for the reduction of C to X, such that $M \cap \kappa \in \kappa$; let $\alpha = M \cap \kappa$.

Let $\pi\colon M \to L_\beta$ be the Mostowski collapse. By Lemma 4.4.2, $\alpha \in E$, witnessed by β. The reduction of C to X, and the fact that $A_\alpha = X \cap \alpha$ (Lemma 4.4.5) shows that $C \cap \alpha$ is Δ^0_1-definable in $(L_\alpha; \in, A_\alpha)$. Thus either $C \cap S_\kappa \cap \alpha$ is nonempty, or the construction puts α into S_κ, and so $\alpha \in S_\kappa \cap C$.

Next, we show that S_κ is nowhere stationary. Since $S_\kappa \subseteq E$, it again suffices to consider regular cardinals $\lambda \leq \kappa$, and we use the same club D as above: $D \cap \lambda$ is a club of λ for all regular $\lambda \leq \kappa$, and for all $\alpha \in D$ there is some $M_\alpha \prec L_{\kappa^+}$ with $\alpha = M_\alpha \cap \kappa$. We show that $D \cap S_\kappa = \emptyset$. Let $\alpha \in D$. To verify that $\alpha \notin S$, we may assume that $\alpha \in E$ and is good. Let β witness that $\alpha \in E$, and let $\pi\colon M_\alpha \to L_\gamma$ be the Mostowski collapse. Since L_γ thinks that α is regular (even with respect to sequences definable over L_γ), we have $\gamma \leq \beta$. By Lemma 4.4.5, $B^\gamma_\alpha = \alpha \setminus A^\gamma_\alpha$ (and $A^\gamma_\alpha = X \cap \alpha$). On the other hand, because both A_α and B_α are defined by existential formulas, they are upwards absolute, which implies that $A^\gamma_\alpha \subseteq A^\beta_\alpha = A_\alpha$ and $B^\gamma_\alpha \subseteq B^\beta_\alpha = B_\alpha$. Since α is good, we must have $A_\alpha = A^\gamma_\alpha = X \cap \alpha$.

Let C be a club of α which is Δ^0_1-definable over $(L_\alpha; \in, A_\alpha)$. Using the same definition over L_κ with X replacing A_α results in a club $\tilde{C}$ of κ which is X-computable and such that $\pi(\tilde{C}) = \tilde{C} \cap \alpha = C$. Now $S_\kappa \in M_\alpha$ and $\pi(S_\kappa) = S_\kappa \cap \alpha$; as $S_\kappa \cap \tilde{C} \neq \emptyset$ and $M_\alpha \prec L_{\kappa^+}$, as above we have $C \cap (S_\kappa \cap \alpha) \neq \emptyset$. So our instructions ensure that $\alpha \notin S$.

It follows that for all $t < \kappa$, S_t is nowhere stationary as well. Since $S_t \subseteq E_t \subseteq E$, it again suffices to check that $S_t \cap \lambda$ is nonstationary in λ, when $\lambda \leq \kappa$ is regular. Now, if $\lambda > t$ then $S_t \subseteq t$ is bounded in λ, and so nonstationary. If $\lambda \leq t$ then as $E_t \cap \lambda = E \cap \lambda$, we can see that $S_t \cap \lambda = S_\kappa \cap \lambda$ (by induction on $\alpha < \lambda$ we see that $S_t \cap \alpha = S_\kappa \cap \alpha$). And we have already observed that $S_\kappa \cap \lambda$ is nonstationary in λ.

It remains to show that $\langle S_t \rangle_{t \leq \kappa}$ is lexicographically nondecreasing and is continuous. Let $s < t \leq \kappa$. Let $\delta < \kappa$ be such that $S_s \upharpoonright \delta = S_t \upharpoonright \delta$. If $\delta \in S_s$ then $\delta \in E_s$ and so $\delta \in E_t$; the same calculation that put δ into S_s holds for S_t, so $\delta \in S_t$. Hence $S_s \leq S_t$ lexicographically.

Let $t \leq \kappa$ be a limit ordinal. Let $\delta < \kappa$ and $s < t$ such that $S_s \upharpoonright \delta = S_t \upharpoonright \delta$. Suppose that $\delta \in S_t$. So $\delta \in E_t$. Since $E_t = \bigcup_{r<t} E_r$, for sufficiently late r, we have $\delta \in E_r$; if $\delta \in E_r$ and $S_r \upharpoonright \delta = S_t \upharpoonright \delta$ then the same calculation that put δ into S_t also puts δ into S_r. This shows that $S_t = \sup_{r<t} S_r$, and completes the proof of Proposition 4.2.8, and so of Theorem 4.1.1.

4.5 Further Work

Theorem 4.1.1 is optimal: R. Johnston has noticed (see [65]) that every κ-computable free group has a basis which is $\Delta^1_1(L_\kappa)$; namely, the $<_L$-least basis is $\Delta^1_1(L_\kappa)$. However, one can study the general question of the "degree spectra" of bases of κ-computable free groups:

Question 4.5.1. What collections of κ-degrees can be realised as the collection of degrees computing bases for some κ-computable free group?

In [56], the authors show that one cannot code much into bases; depending on κ, the most that can be coded into all bases of some group is either $\emptyset'$ or $\emptyset''$. One can ask, though, in view of Proposition 4.4.4, the following:

Question 4.5.2. Is there a κ-computable free group G such that a κ-degree computes a basis for G if and only if it is not first-order definable over L_κ (not computable from $\emptyset^{(n)}$ for any n)? Can the non-arithmetic degrees (those not below $\emptyset^{(\alpha)}$ for any $\alpha < \kappa$) be similarly realised?

Part II

Fractal Geometry and Randomness

Chapter 5

History

We apply effective tools to answer a classical problem of fractal geometry. First, we introduce fractal geometry and algorithmic randomness, as well as their (shared) history and connection. In the final chapter 6 we then use this connection to solve a classical problem of fractal geometry concerning the Hausdorff dimension of projections of plane sets: we show that a theorem due to John Marstrand which describes the behaviour of Hausdorff dimension of projections of $\underset{\sim}{\mathbf{\Sigma}}^1_1(\mathbb{R}^n)$ sets onto straight lines is optimal. In particular, assuming $V{=}L$, we construct $\underset{\sim}{\mathbf{\Pi}}^1_1(\mathbb{R}^2)$ "counterexamples": sets failing the conclusions of Marstrand's theorem.

5.1 Fractal Geometry

Fractal geometry can be considered the study of "rough" subsets of Euclidean space $\mathbb{R}^n$. The tools of this study are measures of complexity: Lebesgue measure springs to mind, and so does the finer Hausdorff measure: while the middle-third Cantor set $\mathcal{C}$ has famously n-dimensional Lebesgue measure 0 for all $n < \omega$ (of course, only the case $n = 1$ is informative here), there exists $s \in \mathbb{R}$ such that the s-dimensional Hausdorff measure of $\mathcal{C}$ is non-zero; in fact it equals 1 [37, Thm 1.14]. Further, this s is the unique real number for which the Hausdorff measure of $\mathcal{C}$ is non-zero and finite—it is called the **Hausdorff dimension** of $\mathcal{C}$. In a rather restrictive sense, fractal geometry can be considered the study of dimension, Hausdorff and otherwise—thus, the roughness of a subset of Euclidean space is characterised by its dimension.

It was Hausdorff's insight in 1918 to tweak Lebesgue measure to obtain a measure on $\mathbb{R}^n$ defined on all subsets of $\mathbb{R}^n$ [58]. The resultant Hausdorff measure together with its associated Hausdorff dimension yield a natural notion of complexity of *any* subset of $\mathbb{R}^n$. Firstly, Hausdorff measure extends Lebesgue measure wherever both are defined. Secondly, Hausdorff dimension behaves naturally (cf. section 5.1.1), and Hausdorff dimension exceeds topological dimension [29, Thm 6.3.10]. Further, the Hausdorff dimension of a set also informs on some of its topological properties. E.g. if A has Hausdorff dimension $s \in (0, 1)$ and its s-dimensional Hausdorff measure is non-zero finite, then A is totally disconnected [37, Lem 4.1]. However, the desirable properties of Hausdorff measure come at a price: computing the Hausdorff dimension for an arbitrary subset of $\mathbb{R}^n$ is often difficult. This difficulty exists to this day, even though novel approaches using computability theory have provided new handles—these will be essential to our work below.

The field we call fractal geometry today was born with the development of **geometric measure theory**. This theory is the result of combining Lebesgue's work [82], and his eponymous measure, with differential algebra: topology, manifolds, exterior algebras and differential forms, Lebesgue and Hausdorff measure, integration theory, as well as the ideas of Borel and

Souslin in descriptive set theory play a role [39]. Much of the early work in unifying known results in the investigation of pathological subsets of $\mathbb{R}^n$ goes back to Abram Besicovitch, whose "pioneering genius" uncovered a "pattern of structure" [39, p. 2], whose theory we now call geometric measure theory.[1]

While the distinction between geometric measure theory and fractal geometry is blurry, one can argue that in its study of pathological subsets of $\mathbb{R}^n$, fractal geometry focusses on complexities. The measures used to capture suitable notions of complexity are those assigning a type of **dimension**. For instance, Falconer's "The geometry of fractal sets" [37] solely focusses on sets of certain Hausdorff dimensions, and classifies their structural properties. As it turns out, those sets of non-integral Hausdorff dimension (i.e. those $A \subset \mathbb{R}^n$ for which $\dim_H(A) \notin \omega$) are pathological in a local sense: relative to a suitable notion of density, their small-scale behaviour is complex [37, p. 20 and Thms 4.11, 4.12]. This fact precisely captures the intuitive notion of **self-similarity**, which is often attributed to fractals.

We give an example of a classical result in fractal geometry which will be of interest to us in future sections. As a result of the naturalness of Hausdorff dimension as a measure of complexity of sets, the investigation of Hausdorff dimension under "natural" transformations became a subject of interest. One class of fundamental non-trivial transformations to be considered was that of orthogonal projections. An early theorem connecting geometric measure theory with fractal geometry is due to John Marstrand, a student of Besicovitch's: in 1954, one year before being conferred his PhD degree at the University of Oxford, Marstrand published [102], which con-

[1] Besicovitch's impact is not least witnessed by the fact that classical Hausdorff dimension is also sometimes called *Hausdorff-Besicovitch dimension*.

tains the now seminal Projection Theorem 6.1.1. Notably, his results predate the coinage of the term *fractal geometry.* Despite its strength, Marstrand's projection theorem only received little attention for a quarter of a century [35]—nowadays, the study of the relationship between Hausdorff dimension and projections is a vibrant research area [108, 61, 126]; we recommend the survey [107] for more on projection theorems. For more details on the history of Marstrand's theorem, we also recommend [35].

The dimensional study of pathological sets also relates to other mathematical fields—and its geometric measure-theoretic investigation has yielded perhaps surprising advances. An example is the **Kakeya problem**, due to Soichi Kakeya and Matsusaburo Fujiwara, which asks [67, 46]: in $\mathbb{R}^2$, what is the minimum area required to continuously rotate a needle of length 1 by π radians and return it to its starting point in reversed position? It is easily seen that a circle of radius $\frac{1}{2}$ does the trick—hence an upper bound for the two-dimensional Kakeya problem is $\frac{\pi}{4}$. Julius Pál [128] showed that if the witnessing set is required to be convex, then the answer is $\frac{1}{\sqrt{3}}$, witnessed by the equilateral triangle of height 1, confirming a conjecture of Fujiwara (see also [42]).[2] Kakeya's problem on $\mathbb{R}^2$ was solved by Besicovitch who realised that finding the minimal area of a set containing a unit line segment in every direction suffices (such sets are nowadays called **Kakeya** (or **Besicovitch**) **sets** [37, 7.2]): he showed that for every $\epsilon > 0$ there exists a set of Lebesgue measure less than ϵ permitting the needle rotation [6].[3]

Foreshadowing our work in subsequent chapters, we mention here that

[2]Further topological considerations, such as what happens if the set is required to be simply connected, have led to various formulations of the Kakeya problem; cf. [21].

[3]Further, Besicovitch realised decades later that his solution to the Kakeya problem was closely related to his previous work in geometric measure theory [7]; see also [37, 7.1].

recent advances in characterising fractal geometry in terms of algorithmic randomness has provided new handles on determining the fractal properties of Kakeya sets, which have garnered interest in their own right. For example, Roy O. Davies [22] showed that every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2; his proof was recently recovered via modern techniques in computability theory by Jack Lutz and Neil Lutz [91]. The general question of whether this is also true for n-dimensional Kakeya sets is known as the *Kakeya set conjecture*. Its higher dimensional versions turn out to be connected to difficult problems in harmonic analysis [156, 69].

Of course, one cannot introduce fractal geometry without mentioning Benoit Mandelbrot, who coined the term **fractal** [98]. Subsequently, Mandelbrot has described empirical connections between self-similar structures in nature and science [99]. This brings us to the problem of definitions—how exactly are fractal sets characterised? Given its history and distinct motivations, no formal universally accepted characterisation of fractals has so far appeared. Instead, and rather vaguely, a set is a fractal “if it has interesting properties related to some notion of dimension”.[4] This view is in particular shared by Kenneth Falconer, who argues there is no intrinsic need to find a universal definition in the first place [34, p. xxv]. For more on the history of fractal geometry, we recommend the Introduction of [37], which was also useful in the compilation of our outline above.

5.1.1 Hausdorff Measure and Dimension

Consider a subset $E \subset \mathbb{R}^2$. To define Hausdorff measure, first consider

[4] E.g. [29] contains at least three different classical “definitions” of *fractal sets*, all based on their dimension properties, Hausdorff, packing, topological, or otherwise.

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i<\omega} |U_i|^s \;\middle|\; E \subset \bigcup_{i<\omega} U_i \wedge (\forall i<\omega)(|U_i|<\delta)\right\}$$

where d is the usual Euclidean distance and $|U| = \sup\{d(x,y) \mid x, y \in U\}$, the *diameter of* U. As δ increases we include more covers in our infimum; thus, if $0 < \delta < \delta'$ then $\mathcal{H}^s_{\delta'}(E) < \mathcal{H}^s_\delta(E)$. In particular, $\lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E)$ always exists (however it might be infinite).

Definition 5.1.1. *Let $E \subset \mathbb{R}^2$. We define the s-**dimensional Hausdorff measure of** E as follows:*

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E).$$

It is easily seen that there must exist a *critical value* for s at which $\mathcal{H}^s(E)$ changes from ∞ to 0—this is the *Hausdorff dimension*.

Definition 5.1.2. *Let $E \subset \mathbb{R}^2$. Then its* **Hausdorff dimension** *is:*

$$\dim_H(E) = \sup\{s \geq 0 \mid \mathcal{H}^s(E) = \infty\} = \inf\{s \geq 0 \mid \mathcal{H}^s(E) = 0\}$$

Hausdorff dimension is well-behaved under Lipschitz maps. Recall that a map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is **Lipschitz with constant** $M > 0$ if for all $x, y \in \mathbb{R}^m$ we have $|f(x) - f(y)| \leq M|x - y|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$. Note that Lipschitz maps cannot increase dimension:

Lemma 5.1.3. *Let $E \subset \mathbb{R}^2$. If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies a Lipschitz condition then $\dim_H(f(E)) \leq \dim_H(E)$.*

We use the following simple lemma.

Lemma 5.1.4. *Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is Lipschitz with constant $M > 0$. Then for all $s \geq 0$ we have $\mathcal{H}^s(f(E)) \leq M^s \mathcal{H}^s(E)$.*

Proof. Suppose f has Lipschitz constant M, and that (U_i) is a δ-cover for E. Then $(f(U_i))$ is an $M\delta$-cover for $f(E)$. Thus

$$\mathcal{H}^s_{M\delta}(f(E)) \leq \sum_{i<\omega} |f(U_i)|^s \leq \sum_{i<\omega} M^s |U_i|^s$$

since, for any $f(x), f(y) \in f(U_i)$ we have $|f(x) - f(y)| \leq M|x - y|$, whence $|f(U_i)| \leq M|U_i|$ follows after taking suprema. Take $\lim_{\delta \to 0^+}$ to finish. □

Proof of Lemma 5.1.3. Suppose $\dim_H(E) = s$, and assume f is Lipschitz with constant $M > 0$. Then $\mathcal{H}^s(f(E)) \leq M^s \mathcal{H}^s(E) < \infty$. Now, $\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$ implies $\mathcal{H}^s(f(E)) < \infty$, so $\dim_H(f(E)) \leq s$. □

In particular, if f is an isometry then Hausdorff measure is fixed:

Corollary 5.1.5. *Hausdorff dimension is preserved under isometries. In particular, it is preserved under rotation and translation.*

5.2 Algorithmic Randomness

Algorithmic randomness is the study of complexity of infinite binary sequences. For its history, we recommend the extensive [27, 86], as well as the chapter "Martingales in the Study of Randomness" in [111]. We use the former to describe outline the milestones of the subject's history, but recommend the reader to consult [27, II.6] for an excellent historical overview.

One of the earliest recorded formalisations of randomness goes back to Richard von Mises [151], who attempted to characterise randomness in terms of rare behaviour. In particular, a real $f \in 2^\omega$ that is random ought to satisfy the law of large numbers:

$$\lim_{n\to\infty} \frac{\sum_{i<n} f(i)}{n} = \frac{1}{2}.$$

However, any real that is empirically random ought to satisfy the law of large numbers on some class of its subsequences as well. Determining this class turned out to be the major hurdle in defining algorithmic randomness correctly, relative to our empirical understanding of randomness. Formally, assume we define $\mathcal{A} \subset \omega^\omega$ with the goal that, for every $f \in 2^\omega$ we have

$$f \text{ is random } \iff (\forall g \in \mathcal{A})(f^g \text{ satisfies the law of large numbers})$$

where f^g denotes the subsequence of f determined by g. Firstly, it is clear that if f contains infinitely many zeros, then we can always define a subsequence that is the constant zero function. Hence, if we required a random string to satisfy the law of large numbers on *every* subsequence (i.e. if $\mathcal{A} = \omega^\omega$), then *no* random sequences existed. A bottom-up approach appears reasonable then: choose an $\mathcal{A}_0$ and close it up[5] whenever we encounter an empirically random string not captured by $\mathcal{A}_0$.

This is precisely the approach Alonzo Church [19] and Jean Ville [153] took. In his [151, p. 58], von Mises motivated his definition in terms of games. In particular, he observed that a random sequence should not admit a *weak winning strategy*: given $f \in 2^\omega$, there should not exist a "Spielsystem"[6] which allows a player to consciously choose a $g \in \mathcal{A}$ for which f^g belongs to some winning set. (E.g. if the winning set is $\{111\cdots\}$, then no random f should admit a strategy g for which $f^g = 111\cdots$; this agrees with our tentative definition above.) Church recognised that von Mises' class of *acceptable Spielsysteme* was too broad. In [19, p. 133] he attempts to relate them to computability theory:

[5]Compare with section 3.2.1, where we considered strengthening notions of independence in free abelian groups

[6]German for "game strategy"

> It may be held that the representation of a *Spielsystem* by an arbitrary function ϕ is too broad. To a player who would beat the wheel at roulette a system is unusable which corresponds to a mathematical function known to exist but not given by explicit definition; and even the explicit definition is of no use unless it provides a means of calculating the particular values of the function. As a less frivolous example, the scientist concerned with making predictions or probable predictions of some phenomenon must employ an effectively calculable function: if the law of the phenomenon is not approximable by such a function, prediction is impossible. Thus a *Spielsystem* should be represented mathematically, not as a function, or even as a definition of a function, but as an effective algorithm for the calculation of the values of a function.

This led Church to define what is nowadays known as **Church** or **computable randomness**, by choosing $\mathcal{A}$ to be the class of computable increasing functions. It was Ville [153] who realised that Church's $\mathcal{A}$ was not adequate either. For he observed that one could construct a real f for which f^g satisfies the law of large numbers for all g in Church's $\mathcal{A}$, yet for all n

$$\frac{\sum_{i<n} f(i)}{n} \leq \frac{1}{2}.$$

Of course, this is not the type of behaviour expected from a random sequence; instead, an empirically random sequence should hover around $\frac{1}{2}$ in its initial proportions. So, Ville proposed to extend Church's $\mathcal{A}$ so that it also captures Ville's sequence. But similar to characterising free abelian groups via independence (cf. section 3.2.1), it is not at all obvious why this larger class should be *complete* with respect to capturing randomness, either.

A roadblock was hit, and it transpired that the bottom-up approach of constructing $\mathcal{A}$ might not succeed. Indeed, one way to correctly define algorithmic randomness arrived as a top-down argument: Per Martin-Löf realised that the complete extension of $\mathcal{A}$ ought to describe reals that have no rare properties [105]. This agrees with Church's approach: any algorithm describing a non-random string can be given by a Turing machine (via the Church-Turing-thesis), and as there are only countably many such machines most reals ought to be random empirically. Martin-Löf's insight was that the rareness of properties should not only be empirically observed by measure-theoretical arguments, but in fact characterised by them. His work resulted in an important class of statistical tests which comprise all reasonable tests (such as the law of large numbers)—these are called **Martin-Löf tests**, and form a cornerstone of the theory today.

At roughly the same time, two further approaches to randomness were developed independently: Claus-Peter Schnorr [136] developed a notion of randomness based on **left-c.e. martingales**, carrying von Mises' original idea of randomness in terms of game theory forward: a real is random if and only if no such martingale succeeds on it. In game theoretic terms, f is random if there is no c.e. strategy on the n-th bit of f given $f \restriction n$ with which one could be guaranteed to make infinite profit.

At the same time, Andrey Kolmogorov [76] (in the Soviet Union) and Ray Solomonoff [139] (in the United States) independently developed the early theory of **algorithmic complexity theory**, focussing on a notion of complexity for strings (rather than infinite sequences). At its core lies the question: how *difficult* is it to describe finite binary string? Complexity is measured by the length of the shortest Turing program computing said string. Their work on *Kolmogorov complexity* was later extended by Leonid Levin [84] and Gregory Chaitin [14] to a more well-behaved theory: that of **prefix-free complexity**, generally denoted by K. Here, the program computing the binary string must be **prefix-free**. This K extends nicely to infinite sequences: a real $f \in 2^\omega$ is **Kolmogorov random** (or **Chaitin random**) if each of its initial segments is K-difficult to compute.

The next theorem proves most convincingly the empirical correctness of these randomness notions.

Theorem 5.2.1 ([27, Thm 6.2.3, 6.3.4])**.** *Let $f \in 2^\omega$. T.f.a.e:*

1. *f is* **Kolmogorov random:** *there exists $c < \omega$ such that $K(f \restriction n) \geq n - c$ for all $n < \omega$.*
2. *f is* **Martin-Löf random:** *f passes every Martin-Löf test.*
3. *f is* **martingale random:** *no left-c.e. martingale succeeds on f.*

5.2.1 Kolmogorov Complexity

Very comprehensive references for Kolmogorov, or *prefix-free*, complexity include Downey and Hirschfeldt [27] and Li and Vitány [86]. A very short but very readable introduction is Fortnow [43]. We give a very brief introduction.

We choose to express the basic notions of algorithmic information theory using Turing machines. We pick a *universal prefix-free* machine as our *reference machine U*. Such a machine exists by construction: every prefix-free p.c. function has a prefix-free machine that computes it (e.g. [27, 3.5]). Our universal machine U satisfies that every prefix-free p.c. function $f\colon 2^{<\omega} \to 2^{<\omega}$ has a program p_f for which $U(p_f, x) = f(x)$. Define h such that whenever

p_f is a prefix-free program for a prefix-free p.c. function f then

$$h\left(0^{|p_f|}1p_f x\right) = U(p_f, x) = f(x).$$

Observe that h is p.c. and prefix-free. If $A \in 2^\omega$ is an oracle, let U^A be the universal prefix-free machine that has access to the oracle A (this means, the machine can perform a step of the type "does k belong to A?" for any $k < \omega$, and branch accordingly), and define h_A analogously.

If $\sigma \in 2^{<\omega}$ let $l(\sigma)$ denote the length of σ.

Definition 5.2.2. *Let $\sigma \in 2^{<\omega}$. The* **Kolmogorov complexity of** *σ is*

$$K(\sigma) = \min\{l(\rho) \mid h(\rho) = \sigma\}.$$

If A is an oracle, $K^A(\sigma)$ is defined analogously, with h_A in place of h.

All log are $\log_2$. As a result of prefix-freeness we immediately obtain:

Lemma 5.2.3. *If $\sigma, \tau \in 2^{<\omega}$ then $K(\sigma) \leq^+ l(\sigma) + 2\log(l(\sigma))$, and $K(\sigma\tau) \leq^+ K(\sigma) + K(\tau)$*

In the upcoming chapter, we focus on the construction of reals in terms of Kolmogorov complexity: via prefix-free complexity, there exist finer classifications of randomness on reals in 2^ω than the "binary random/non-random" distinction from Theorem 5.2.1—hence we can construct reals of any complexity $\epsilon \in [0, 1]$, an important fact when combining the theory with fractal geometry (and the notion of Hausdorff dimension).

5.3 Effective Notions in Fractal Geometry

The field of effective dimension connects measures of complexity of reals with the dimension of sets. This connection was investigated by Ryabko [132, 133] in the 1980s, and by Staiger [146] and Cai and Hartmanis [10] in the 1990s, who explored the relationship between Kolmogorov complexity, martingales, and Hausdorff dimension of reals explicitly. The theory gained traction with Jack Lutz' papers on *constructive dimension* of classes of reals [89]. His framework allowed a meaningful assignment of complexity to both classes of reals *and* to reals themselves [90]. Since computable reals have constructive dimension 0, and Martin-Löf randoms have constructive dimension 1 (the maximal possible), ML-randomness cannot distinguish non-computable non-random reals. Lutz' framework can capture these finer classifications. In this work, Lutz generalised martingales to game-theoretic maps called **(term) gales** [88, 89]. This interaction between probabilistic tools in the context of randomness highlights the connections established decades prior (cf. Theorem 5.2.1). In hindsight, it also should not be surprising that constructive dimension can be characterised in terms of Kolmogorov complexity of its initial segments: constructive dimension is determined by the **algorithmic information content** of reals; this discovery is due to Elvira Mayordomo [110]. Hitchcock then isolated a correspondence principle between K and $\dim_H$ on unions of Π^0_1 subsets of 2^ω [60, Cor 4.3] (also see [59]).

In 2000, Jack Lutz had already hinted at a deep connection between his constructive dimension and classical Hausdorff dimension[7]. His hunch turned out to be true: the breakthrough came with his and N. Lutz' **point-to-set principle** 5.3.3, which characterises the Hausdorff dimension of sets of reals in terms of the constructive dimension of their points [91].

[7]Hitchcock remarks that "Lutz conjectured that there should be a correspondence principle stating that the constructive dimension of every sufficiently simple set X coincides with its classical Hausdorff dimension" in lectures at Iowa State university [60, p. 559].

Definition 5.3.1. *Let $f \in 2^\omega$. Define the* **(effective) dimension of** f *by*

$$\dim(f) = \liminf_{r\to\infty} \frac{K(f \upharpoonright r)}{r}$$

This relativises: if $A \in 2^\omega$ is an oracle then define

$$\dim^A(f) = \liminf_{r\to\infty} \frac{K^A(f \upharpoonright r)}{r}.$$

This notion can be naturally extended to Euclidean space:

Definition 5.3.2. *Let $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Then we define the* **Kolmogorov complexity of** x **at precision** $t < \omega$ *by*

$$K_t(x) = \min\{K(q) \mid q \in \mathbb{Q}^m \cap B_{2^{-t}}(x)\}$$

where $B_s(y)$ is the open ball with respect to the Euclidean metric, with radius s and centre y. The effective Hausdorff dimension *of x is given by*

$$\dim(x) = \liminf_{t\to\infty} \frac{K_t(x)}{t}.$$

The characterisation of effective Hausdorff dimension of reals given in Definition 5.3.2 is due to Mayordomo [110].[8] We now state Jack Lutz' and Neil Lutz' point-to-set principle, which we use in our subsequent arguments.

Theorem 5.3.3 (Point-to-set Principle, [91, Thm 1])**.** *Let $n < \omega$ and $E \subset \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A\in 2^\omega} \sup_{x\in E} \dim^A(x).$$

Lutz and Lutz [91] and Lutz and Stull [95, 96] provide applications of the point-to-set principle, which has also been extended to arbitrary separable metric spaces [92].

[8] Further, replacing lim inf by lim sup in Definition 5.3.2 yields a characterisation of effective packing dimension [93], which is denoted by $\mathrm{Dim}(x)$.

Chapter 6

Co-analytic Counterexamples to Marstrand's Projection Theorem

The results in this chapter concern Marstrand's Projection Theorem, a seminal theorem of classical fractal geometry. Below we give an introduction to the theorem and its history. We then move on to consider sets that fail Marstrand's Theorem—the first one to construct such a "counterexample" being Roy O. Davies [23]. In that vein we consider the complexity of possible definable counterexamples. Finally, under the set-theoretic assumption $V{=}L$ we show that there exist co-analytic counterexamples to Marstrand's Projection Theorem. It follows from Marstrand's original theorem that this complexity is optimal.

This chapter is based on [131], which has been submitted for publication. The results in this chapter have independently been obtained by T. Slaman and D. Stull.

6.1 The Complexity of Projections

Since the early development of fractal geometry, a few theorems have turned out to be foundational. **Marstrand's projection theorem** [102] dating back to 1954 is one of them. While ignored for decades (with the term "fractal geometry" only arriving in the 1970s), fractal geometry, and projection theorems like Marstrand's, are researched intensively nowadays [35].

The theorem states that orthogonal projections of $\utilde{\mathbf{\Sigma}}^1_1$ sets cannot drop too far in dimension—but there are exceptions. R. O. Davies [23] constructed a counterexample non-constructively, assuming CH. Simplifications [70] and generalisations [106] of Marstrand's theorem followed over time. Nowadays, the standard argument to prove Marstrand's projection theorem is Kaufman's proof [70] based on *energy potential characterisations* of Hausdorff dimension. Refinements are sought after today [5, 36].

Here, we focus on the possible complexity of Marstrand "counterexamples": sets failing the conclusion of Marstrand's theorem. In particular, we show that being $\utilde{\mathbf{\Sigma}}^1_1$ is sharp for Marstrand's theorem (a fact previously unknown), by constructing $\utilde{\mathbf{\Pi}}^1_1$ counterexamples to Marstrand's projection theorem. In particular, we use the point-to-set principle 5.3.3 to construct a set of Hausdorff dimension 1, all of whose projections have Hausdorff dimension 0. Using Vidnyánszky's Theorem 2.3.4, the constructed set is $\utilde{\mathbf{\Pi}}^1_1$. We then extend our result in a strong way: for each $\epsilon \in (0, 1)$, we produce a $\utilde{\mathbf{\Pi}}^1_1$ set X for which $\dim_H(X) = 1 + \epsilon$ while its projection onto every line through the origin has dimension ϵ, the minimal allowable value.

Note that we prove *consistency*: we make the set-theoretic assumption that $V{=}L$, which ensures co-analycity via Vidnyánszky's theorem.

If $E \subset \mathbb{R}^2$, let $\mathrm{proj}_\theta(E)$ denote its projection onto the unique line passing through the origin at angle θ with the first axis. Let μ denote the one-dimensional Lebesgue measure.

Theorem 6.1.1 ([102, Thm I & II]). *Let $E \subset \mathbb{R}^2$ be $\underset{\sim}{\mathbf{\Sigma}}^1_1(\mathbb{R})$. For almost all $\theta \in [0, \pi)$:*

1. *If $\dim_H(E) \leq 1$ then $\dim_H(\mathrm{proj}_\theta(E)) = \dim_H(E)$.*
2. *If $\dim_H(E) > 1$ then $\mu(\mathrm{proj}_\theta(E)) > 0$.*

Pertti Mattila has extended the result to $\mathbb{R}^n$ as follows [106]: suppose $0 < m < n$. If $s \in (m, n]$ and E has non-zero finite s-dimensional Hausdorff measure, then $p(E)$ has positive m-dimensional Lebesgue measure for almost all orthogonal projections p; further, if $s \in [0, m]$ then, again for almost all orthogonal projections p, we have $\dim_H(p(E)) = \dim_H(E)$.

Note that item 2 is strictly stronger than an assertion about Hausdorff dimension [5]. This follows from the fact that Hausdorff measure generalises m-dimensional Lebesgue measure on $\mathbb{R}^m$ [29, 6.1]. In particular, if $A \subset \mathbb{R}$ has positive Lebesgue measure, then $\dim_H(A) = 1$. Hence item 2 implies:

Corollary 6.1.2. *If $E \subset \mathbb{R}^2$ is analytic and $\dim_H(E) > 1$ then for almost all $\theta \in [0, \pi)$ we have $\dim_H(\mathrm{proj}_\theta(E)) = 1$.*

Beyond geometric measure-theoretical approaches, the point-to-set principle 5.3.3 has proven to be very useful in investigating orthogonal projections. This applies to both Hausdorff dimension $\dim_H$ and **packing dimension** $\dim_P$, a *dual*[1] to Hausdorff dimension due to Claude Tricot [150]: N. Lutz and Stull [95] have shown using algorithmic arguments that if $X \subset \mathbb{R}^2$

[1] Cf. Theorems 5.3.3 and 6.6.2.

satisfies $\dim_H(X) = \dim_P(X)$ then Marstrand's theorem applies[2]. They also give a new bound on the packing dimension of orthogonal projections under packing dimension, and provide a new proof of Marstrand's theorem. Ted Slaman has shown that under $V{=}L$ there exists $A \subset \mathbb{R}$ of Hausdorff dimension 1, all of whose closed subsets are countable (and hence trivially of Hausdorff dimension 0) [138], also using the point-to-set principle—in said work, the author investigates the capacitability of sets, a type of regularity of its closed subsets (the witness is the set of self-constructibles $\mathcal{C}_1$).

6.1.1 Our Theorems

We provide co-analytic counterexamples to both items 1 and 2 of Theorem 6.1.1. Since analytic sets satisfy Theorem 6.1.1, our results are sharp. The first theorem is proven in section 6.4, the second in section 6.5. We emphasise that our results are both theorems of $\mathsf{ZF} + V{=}L$.

Theorem 6.4.1 ($V{=}L$)**.** *There exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while, for every $\theta \in [0, 2\pi)$ we have $\dim_H(\text{proj}_\theta(E)) = 0$.*

Theorem 6.5.1 ($V{=}L$)**.** *For every $\epsilon \in (0, 1)$, there exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1 + \epsilon$ while, for every $\theta \in [0, 2\pi)$ we have $\dim_H(\text{proj}_\theta(E)) = \epsilon$.*

The constructions in both proofs use Vidnyánszky Theorem 2.3.4, which allows us to build co-analytic sets recursively (recall section 2.3 for details). The arguments in the proof of Theorem 6.5.1 are similar to those in Theo-

[2] This is not a characterisation; there exist non-analytic sets for which Hausdorff and packing dimension agree.

rem 6.4.1; in fact, Theorem 6.5.1 subsumes Theorem 6.4.1. However, as the argument in Theorem 6.4.1 is less involved, we present it separately.

Note. In our theorems there are no "good" angles. This contrasts with Marstrand's Theorem 6.1.1, which asserts that for any analytic set E the set of "bad" angles θ—for which the projection $\text{proj}_\theta(E)$ does *not* have maximal Hausdorff dimension—is a null set. On the other hand, the "counterexamples" we construct fail Marstrand's theorem for *every* angle: the set of "good" angles—for which the projection *does* have maximal Hausdorff dimension—is not only null, but empty.

6.2 Coding Objects by Finite Strings

While, formally, our arguments take place in $2^{<\omega}$, we naturally identify certain finite strings with objects in the domain of discourse; these are usually rational numbers (elements of $\mathbb{Q}$) and natural numbers (elements of ω). As is common, this identification takes place in the meta-theory; however, determining whether a string is to be identified as e.g. a rational is computable. We normally denote the string representation using an overline: if x is an object in the domain of discourse, then $\overline{x}$ denotes the string identifying it.

Fixing a particular coding is illustrative. We code objects in the domain of discourse as follows (the implied operation on finite strings is concatenation):

- If $k < \omega$ then let $\overline{k}$ be the string whose digits are given by the binary expansion of k.
- If $n \in \mathbb{Z}$ then let w be the binary expansion of n with each digit doubled ($n = 101$ becomes $w = 110011$). Then let $\overline{n} = w01$ if $n \geq 0$, and $\overline{n} = w10$ otherwise.

- If $q \in \mathbb{Q}$ then suppose $q = a/b$. Then let $\overline{q} = \overline{a}\overline{b}$.
- If $q = (q_1, \ldots, q_m) \in \mathbb{Q}^m$ then let $\overline{q} = \overline{q_1} \cdots \overline{q_m}$.
- If $x \in \mathbb{R}$, suppose $k < \omega$, and express x in binary. Take the integer part of x and double each digit; denote this string by w. Take the first k bits of x after the binary point, denoted by z. If $x \geq 0$, let $\overline{x}[k] = w01z$; otherwise define $\overline{x}[k] = w10z$.
- If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, suppose $k < \omega$. Then let $\overline{x}[k] = \overline{x_1}[k] \cdots \overline{x_m}[k]$.
- If $x \in \mathbb{R}$ then let $\overline{x} \in 2^\omega$ be the limit of $\overline{x}[k]$ in the obvious fashion. If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ then interweave $\overline{x_1}, \ldots, \overline{x_m}$ bit by bit.

Observe that, using this coding[3], if $k < \omega$ then $l\left(\overline{k}\right) \leq \log(k) + 1$.

The distinction between strings and objects is particularly important when we discuss real numbers (i.e. objects in $\mathbb{R}$), and their truncated approximations. In other cases we are more casual; for instance, we normally write $K(k)$ and $K(q)$ instead of the formally correct $K(\overline{k})$ and $K(\overline{q})$.

The following technical lemma allows us to work with finite strings instead of approximating rationals when considering Kolmogorov complexity.

Lemma 6.2.1 ([96, Cor 2.4]). *For every $m < \omega$ there exists a constant c such that for all $t < \omega$ and $x \in \mathbb{R}^m$ we have*

$$|K_t(x) - K(\overline{x}[t])| \leq K(t) + c.$$

The proof uses the fact that $\overline{x}[t]$ gives a reasonable approximation to x, in the sense that its distance to x is bounded by a function that only depends on m. In Proposition 6.3.1, we provide a similar identification argument for polar coordinates. We note an important corollary right here.

[3]These codings are not necessarily optimal.

Corollary 6.2.2. *If $m \geq 1$ and $x \in \mathbb{R}^m$ then*

$$\dim(x) = \liminf_{r\to\infty} \frac{K(\overline{x}[r])}{r}.$$

6.3 Arguing in Polar Coordinates

In the course of our constructions in both Theorems 6.4.1 and 6.5.1, it will be easier to work in polar coordinates than in Euclidean coordinates. A point (x, y) in Euclidean space *has polar coordinates* (r, θ) if and only if $x = r\cos\theta$ and $y = r\sin\theta$. We will restrict our attention to the first quadrant of the unit disc, which we denote by

$$\mathbb{D} = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y \geq 0 \wedge \sqrt{x^2 + y^2} \leq 1\right\}.$$

Thus $r \in [0, 1]$ and $\theta \in [0, \pi/2]$. Importantly, all points expressed in the proofs below are given in Euclidean coordinates. When we write (r, θ) we do *not* mean $(r\cos\theta, r\sin\theta)$.

The following proposition can be considered an analogue to Lemma 6.2.1; its proof follows the ideas of the proof of Lemma 6.2.1 from [96, Cor 2.4].

Proposition 6.3.1. *Suppose $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ). Then*

$$\dim(x, y) = \dim(r, \theta).$$

We provide proofs to both directions of Proposition 6.3.1 individually below, which also show that the result relativises. We use the following result of Case and J. Lutz:

Lemma 6.3.2 ([12])**.** *There exists a constant c such that for all $m, s, \Delta s < \omega$ and all $x \in \mathbb{R}^m$ we have*

$$K_s(x) \leq K_{s+\Delta s}(x) \leq K_s(x) + K(s) + c_m(\Delta s) + c$$

where $c_m(\Delta s) = K(\Delta s) + m\Delta s + 2\log\left(\left\lceil \frac{1}{2}\log(m)\right\rceil + \Delta s + 3\right) + \left(\left\lceil \frac{1}{2}\log(m)\right\rceil + 3\right)m + K(m) + 2\log(m)$.

Observe that the term $c_m(\Delta s)$ does not depend on s. We also require the following classical lemma.

Lemma 6.3.3 ([8, p. 151])**.** *Suppose $C \subset \mathbb{R}^2$ is compact and convex. If the function $f\colon C \to \mathbb{R}^2$ sending (x,y) to $f(x,y)$ is continuously differentiable on C then it satisfies a Lipschitz condition on C.*

The first halves of our proofs below follow the same argument as [96, Lem 2.3]. Note the map $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is continuously differentiable everywhere, and that $[0,1]\times[0,\pi/2]$ is of course compact and convex.

Lemma 6.3.4 (First half of Proposition 6.3.1)**.** *There exists a constant c such that whenever $(x,y) \in \mathbb{D}$ has polar coordinates (r,θ) then for all $s < \omega$*

$$K_s(x,y) \leq K\left(\overline{r}[s]\overline{\theta}[s]\right) + K(s) + c.$$

Proof. By Lemma 6.3.3, the map translating polar into Cartesian coordinates satisfies a Lipschitz condition as $[0,1]\times[0,\pi/2]$ is compact and convex: there exists $M > 0$ such that if $(r,\theta), (r',\theta') \in [0,1]\times[0,\pi/2]$ then

$$|(r\cos\theta, r\sin\theta) - (r'\cos\theta', r'\sin\theta')| \leq M|(r,\theta) - (r',\theta')|. \tag{$*$}$$

Let $(r,\theta) \in [0,1]\times[0,\pi/2]$, and suppose $(x,y) = (r\cos\theta, r\sin\theta)$.

- Let $r_s = r[s]$, and $\theta_s = \theta[s]$, the truncation of r and θ to s bits after the binary point.

- We will consider the approximation $r_s \cos\theta_s$ of $r\cos\theta$ ($r_s \sin\theta_s$ for $r\sin\theta$ resp.); however, this approximation will in general not yield a finite string; hence we consider truncations to s bits after the binary point:

$$x[s] = (r_s \cos\theta_s)[s] \qquad \text{and} \qquad y[s] = (r_s \sin\theta_s)[s]$$

These truncations allow us to approximate the point (x, y) effectively:

Claim 1. $(x[s], y[s]) \in B_{2^{-s}\left(1+M\sqrt{2}\right)}(x, y)$

Proof of Claim 1. Recall that $x = r\cos\theta$ and $y = r\sin\theta$, and that $x[s] = (r_s \cos\theta_s)[s]$; hence $|(x[s], y[s]) - (r_s\cos\theta_s, r_s\sin\theta_s)| \leq 2^{-s}$, by construction. Using the nomenclature introduced and the Lipschitz condition $(*)$ above we can compute the maximum error when $(x[s], y[s])$ approximates (x, y):

$$\begin{aligned}
|(x[s], y[s]) - (x, y)| &= |(x[s], y[s]) - (r_s\cos\theta_s, r_s\sin\theta_s) \\
&\qquad + (r_s\cos\theta_s, r_s\sin\theta_s) - (x, y)| \\
&\leq |(x[s], y[s]) - (r_s\cos\theta_s, r_s\sin\theta_s)| \\
&\qquad + |(r_s\cos\theta_s, r_s\sin\theta_s) - (x, y)| \\
&\leq 2^{-s} + |(r_s\cos\theta_s, r_s\sin\theta_s) - (r\cos\theta, r\sin\theta)| \\
&\leq 2^{-s} + M|(r_s, \theta_s) - (r, \theta)| \\
&\leq 2^{-s} + M\sqrt{(r - r_s)^2 + (\theta - \theta_s)^2} \\
&< 2^{-s} + M\sqrt{(2)2^{-2s}} \\
&= 2^{-s}(1 + M\sqrt{2})
\end{aligned}$$

$\dashv$

Observe that $2^{-t} = 2^{-s}\left(1 + M\sqrt{2}\right)$ if and only if $t = s - \log\left(1 + M\sqrt{2}\right)$. Therefore, if we can compute $(x[s], y[s])$ then we can compute (x, y) at precision $t = s - \log\left(1 + M\sqrt{2}\right)$. Letting $\Delta t = \log\left(1 + M\sqrt{2}\right)$ we hence have

$$K_{s-\Delta t}(x, y) \leq K(x[s], y[s]) \leq K(\overline{x}[s]\overline{y}[s]) + c'$$

where c' is the machine constant that turns the string representations $\overline{x}[s]$ and $\overline{y}[s]$ into the approximations (i.e. rationals) $x[s]$ and $y[s]$. Now, $t + \Delta t = s$, so the right-hand side of Lemma 6.3.2 implies

$$\begin{aligned} K_s(x,y) &\leq K_{s-\Delta t}(x,y) + K(t) + c_2(\Delta t) + c \\ &\leq K(\overline{x}[s]\overline{y}[s]) + c' + K(t) + c_2(\Delta t) + c \end{aligned}$$

where $c_2(\Delta t)$ is as in Lemma 6.3.2 and hence does not depend on t, and thus not on s. Finally, we prove the following claim:

Claim 2. $K(\overline{x}[s]\overline{y}[s]) \leq K\left(\overline{r}[s]\overline{\theta}[s]\right) + c''$ for some constant c''.

Proof of Claim 2. Recall that $r_s = r[s]$ and $\theta_s = \theta[s]$. Then

$$x[s] = (r_s \cos\theta_s)[s]$$

which (as well as $y[s]$) is computable via approximations and Taylor's Theorem (multiplication and cos are computable, with machine constant c''). ⊣

To recap, we established so far that

$$K_s(x,y) \leq K(\overline{x}[s]\overline{y}[s]) + c' + K(t) + c_2(\Delta t) + c.$$

By the claim we now have

$$\leq K\left(\overline{r}[s]\overline{\theta}[s]\right) + c' + c'' + K(t) + c_2(\Delta t) + c.$$

Recall that $\Delta t = \log\left(1 + M\sqrt{2}\right)$ which is constant. Further, $t = s - \Delta t$, and thus there exists a constant c''' (which only depends on Δt, and hence only on M and not on s) for which $K(t) = K(s - \Delta t) \leq K(s) + c'''$. Hence

$$K_s(x,y) \leq K\left(\overline{r}[s]\overline{\theta}[s]\right) + K(s) + d$$

where $d = c' + c'' + c''' + c_2(\Delta t) + c$ as required. □

For the second half, we make the following brief observation. The argument below will focus on points in $\mathbb{D}$ that do not lie on the first axis; this is necessary for a bounding argument involving Lipschitz conditions: for each point (x, y) not on the first axis, we can find a nice neighbourhood on which the coordinate transformation map from Euclidean to polar coordinates is nicely behaved. What about the points *on* the first axis? There is nothing to do, for if $x \geq 0$, the polar coordinates and Euclidean coordinates of the point $(x, 0)$ coincide. Hence Proposition 6.3.1 holds on the first axis trivially.

Lemma 6.3.5 (Second half of Proposition 6.3.1)**.** *There exists a constant c such that whenever $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) then there exist $N_{(x,y)} < \omega$ and $\Delta < \omega$ such that if $s > N_{(x,y)}$ then*

$$K\left(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]\right) \leq K_s(x, y) + K(s) + c.$$

Proof. First, we make an approximating observation.

Claim 1. For $a \in \mathbb{Q}^2 \cap B_{2^{-r}}(x, y)$ we have $(x[r], y[r]) \in B_{2^{-r}\left(1+\sqrt{2}\right)}(a)$.

Proof of Claim 1. By assumption, $|(x, y) - a| < 2^{-r}$, so by the triangle inequality we have

$$\begin{aligned}
|(x[r], y[r]) - a| &\leq |(x[r], y[r]) - (x, y)| + |(x, y) - a| \\
&= \sqrt{(x[r] - x)^2 + (y[r] - y)^2} + |(x, y) - a| \\
&\leq \sqrt{2(2^{-2r})} + 2^{-r} \\
&\leq 2^{-r}\sqrt{2} + 2^{-r} \\
&= 2^{-r}(1 + \sqrt{2}).
\end{aligned}$$

$\dashv$

In the notation of [96], let $\mathcal{Q}_r^2 = \{2^{-r}z \mid z \in \mathbb{Z}^2\}$ denote the set of r-dyadics. Observe that r-dyadics have at most r-many non-zero post-binary-point bits. It is easy to bound the number of r-dyadics in any open ball:

Claim 2. For any $a \in \mathbb{Q}^2$ and $r < \omega$, we have

$$\left|\mathcal{Q}_r^2 \cap B_{2^{-r}\left(1+\sqrt{2}\right)}(a)\right| \leq \left(4\left(1+\sqrt{2}\right)\right)^2 .$$

Proof of Claim 2. Let C_2 be the square with side length $2\left(1+\sqrt{2}\right)2^{-r}$ that is centred at a. It is clear that $B_{2^{-r}(1+\sqrt{2})} \subset C_2$ and thus

$$\left|\mathcal{Q}_r^2 \cap B_{2^{-r}\left(1+\sqrt{2}\right)}\right| \leq \left|\mathcal{Q}_r^2 \cap C_2\right| .$$

Observe that C_2 has area $\left(2\left(1+\sqrt{2}\right)\right)^2 2^{-2r}$. Now, if $x, y \in \mathcal{Q}_r^2$ and $x \neq y$ then $|x-y| \geq 2^{-r}$ (since the elements in $\mathcal{Q}_r^2$ have at most r-many non-zero post-binary-point bits). Hence consider a *small square*: a square of side length 2^{-r}. Every small square has area 2^{-2r} and cannot contain more than 4 r-dyadics: one on each of its vertices. Hence, dividing the area of C_2 by the area of a small square and multiplying by 4 for each vertex gives an upper bound for the number of r-dyadics:

$$\left|\mathcal{Q}_r^2 \cap B_{2^{-r}\left(1+\sqrt{2}\right)}\right| \leq \frac{\left(2\left(1+\sqrt{2}\right)\right)^2 2^{-2r}}{2^{-2r}}(2^2) = \left(4\left(1+\sqrt{2}\right)\right)^2 \qquad \dashv$$

Let M with program P be a machine that does the following: on input $\pi = \pi_1\pi_2\pi_3$ if $h(\pi_1) = \overline{k}$ with $k < \omega$, and $h(\pi_2) = \overline{t}$ with $t < \omega$, and $h(\pi_3) = \overline{a}$ with $a = (p, q) \in \mathbb{Q}^2$, then M outputs the k-th dyadic rational in $B_{2^{-t}\left(1+\sqrt{2}\right)}(a)$. Suppose $a \in \mathbb{Q}^2$ witnesses the complexity of $K_s(x, y)$; then the claims together imply that $(x[s], y[s])$ is the k-th element in $\mathcal{Q}_r^2 \cap B_{2^{-r}\left(1+\sqrt{2}\right)}(a)$ for some $k < \left(4\left(1+\sqrt{2}\right)\right)^2$. Let the programs π_1, π_2, π_3 be witnesses for $K(k), K(s)$ and $K(a) = K_s(x, y)$, respectively. Then

$$h\left(0^{l(P)}1P\pi_1\pi_2\pi_3\right) = \overline{x}[s]\overline{y}[s]$$

and thus

$$\begin{aligned}K(\overline{x}[s]\overline{y}[s]) &\leq l(\pi_1) + l(\pi_2) + l(\pi_2) + c \\ &= K(k) + K(s) + K_s(x, y) + c' \\ &\leq K_s(x, y) + K(s) + c\end{aligned}$$

where $K(k)$ can be bounded and hence only contributes a constant term.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the computable function mapping a point in Euclidean coordinates to its polar coordinates. On $\mathbb{D}$ (excluding the first axis), this map is given by $(x, y) \mapsto \left(\sqrt{x^2 + y^2}, \tan^{-1}(y/x)\right)$, and is continuously differentiable. Hence take some $\epsilon > 0$ such that the closed ball B of radius ϵ centred at (x, y) does not intersect the first axis. By Lemma 6.3.3, the map f satisfies a Lipschitz condition on B. Now suppose $s < \omega$ is such that $2^{-s} < \epsilon$; thus $B_{2^{-s}}(x, y) \subset B$. Suppose $(p, q) \in \mathbb{Q}^2 \cap B_{2^{-s}}(x, y)$. Recalling that $f(x, y) = (r, \theta)$ we have

$$\begin{aligned}|(r, \theta) - f(p, q)| &= |f(x, y) - f(p, q)| \\ &\leq M|(x, y) - (p, q)| \\ &\leq M2^{-s} \\ &= 2^{-(s - \log M)}.\end{aligned}$$

Thus, computing $(x[s], y[s])$ yields, after applying the machine that computes f with machine constant c'' (compare with claim 2 of the proof of Lemma 6.3.4), the polar coordinates (r, θ) up to precision $s - \log M$:

$$\begin{aligned}K(\overline{r}[s - \log M]\overline{\theta}[s - \log M]) &\leq K(\overline{x}[s]\overline{y}[s]) + c'' \\ &\leq K_s(x, y) + K(s) + c'' + c. \qquad \square\end{aligned}$$

Proof of Proposition 6.3.1. The proof is an easy consequence of the previous two lemmas and the following claim.

Claim 1. If $\Delta < \omega$ then $\left|K\left(\overline{r}[s]\overline{\theta}[s]\right) - K\left(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]\right)\right| \leq c$ for some constant c.

Proof of Claim 1. It is easy to compute $\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]$ from $\overline{r}[s]\overline{\theta}[s]$. For the other direction, let $\overline{r}(\Delta)$ be such that $\overline{r}[s] = \overline{r}[s-\Delta]\overline{r}(\Delta)$, and equally for $\overline{\theta}$. Suppose $h(\pi_1) = \overline{r}[s-\Delta]\overline{\theta}[s-\Delta]$, and $h(\pi_2) = \overline{r}(\Delta)$ and $h(\pi_3) = \overline{\theta}(\Delta)$, and all such programs are optimal. Let p be a program that on input $\pi = \pi_1\pi_2\pi_3$, merges the two strings obtained by π_2 and π_3 with the string from π_1 in the obvious way (recall the coding from section 6.2). Then

$$h\left(0^{l(p)}1p\pi_1\pi_2\pi_3\right) = \overline{r}[s]\overline{\theta}[s]$$

and thus

$$\begin{aligned}K\left(\overline{r}[s]\overline{\theta}[s]\right) &\leq l(\pi_1) + l(\pi_2) + l(\pi_3) + c\\ &= K\left(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]\right) + K(\overline{r}(\Delta)) + K\left(\overline{\theta}(\Delta)\right) + c\end{aligned}$$

by optimality. Observe that $l(\overline{r}(\Delta)) = \Delta$, and recall that $K(\sigma) \leq l(\sigma) + 2\log(l(\sigma)) + c'$ if $\sigma \in 2^{<\omega}$. Since $\Delta < \omega$ is fixed the following suffices:

$$K\left(\overline{r}[s]\overline{\theta}[s]\right) \leq K\left(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta]\right) + 2l\left(\overline{\Delta}\right) + 4\log\left(l\left(\overline{\Delta}\right)\right) + c \qquad \dashv$$

The claim now yields the result from the previous two lemmas: let $(x, y) \in \mathbb{D}$ with polar coordinates (r, θ). Suppose Δ is as in Lemma 6.3.5. Then

$$\begin{aligned}\dim(r,\theta) &= \liminf_{s\to\infty} \frac{K_s(r,\theta)}{s}\\ &= \liminf_{s\to\infty} \frac{K(\overline{r}[s]\overline{\theta}[s])}{s}\\ &= \liminf_{s\to\infty} \frac{K(\overline{r}[s-\Delta]\overline{\theta}[s-\Delta])}{s}\end{aligned}$$

$$= \liminf_{s\to\infty} \frac{K_s(x,y)}{s}$$
$$= \dim(x,y)$$

using the fact that $K(s) \leq \log(s) + 2\log(\log(s)+1) + c$ for some constant. $\square$

We may now pass to polar coordinates as required. In particular, the points of the co-analytic sets we build in Theorems 6.4.1 and 6.5.1 are determined by their radii, which we will construct explicitly. From now on, if we write (r,θ) below, we mean the point that has Euclidean coordinates $(r\cos\theta, r\sin\theta)$; in such cases, $(r,\theta) \in \mathbb{D}$. We occasionally return to Euclidean coordinates, however, and we will explicitly mention when we do so.

6.3.1 Projections in Polar Coordinates

The focus of this part of the present thesis is placed on the dimensional behaviour of projections of points onto straight lines. We make some simple geometric observations below that will simplify arguments later on. Consider $\theta \in [0,\pi]$, and let L_θ be the straight line that passes through the origin at angle θ with the first coordinate axis. It is clear that $[0,\pi]$ exhausts all straight lines through the origin. Let $(s,\rho) \in \mathbb{D}$ and denote by $\operatorname{proj}_\theta(s,\rho)$ the *projection of* (s,ρ) *onto* L_θ: the unique point of intersection of L_θ with the unique perpendicular-to-L_θ line containing (s,ρ). Recall that if $(s,\rho) \in \mathbb{D}$ then $0 \leq s \leq 1$. Compare with figs. 6.1 and 6.2.

There are two cases: either $|\theta-\rho| \leq \pi/2$ or $|\theta-\rho| > \pi/2$. If $|\theta-\rho| \leq \pi/2$ then the length of the projection is given by $|\operatorname{proj}_\theta(s,\rho)| = s\cos(\theta-\rho)$; otherwise $|\operatorname{proj}_\theta(s,\rho)| = s\cos((\theta+\pi)-\rho)$. Since $\cos(x+\pi) = -\cos(x)$ and $0 \leq s \leq 1$, we conclude:

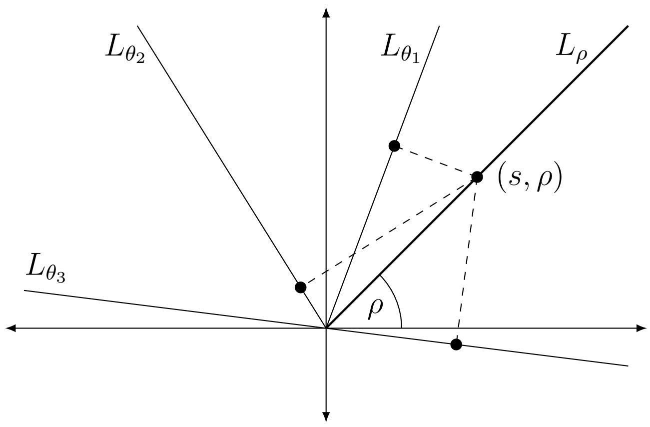

Figure 6.1: With $(s, \rho) \in L_\rho$, the projections onto lines with angles θ_1, θ_2 are straightforwardly obtained from the respective angles.

Lemma 6.3.6. *For every* $(s, \rho) \in \mathbb{D}$ *and every* $\theta \in [0, \pi]$ *we have*

$$|\operatorname{proj}_\theta(s, \rho)| = s|\cos(\theta - \rho)|.$$

In particular, the polar coordinates of the projection of (s, ρ) *onto* L_θ *are*

$$\operatorname{proj}_\theta(s, \rho) = \begin{cases} (s|\cos(\theta - \rho)|, \theta) & \textit{if } |\theta - \rho| \leq \pi/2 \\ (s|\cos(\theta - \rho)|, \theta + \pi) & \textit{otherwise.} \end{cases}$$

Now suppose $E \subset \mathbb{D}$ and fix some $\theta \in [0, \pi]$. Define

$$E(\theta) = \{s|\cos(\theta - \rho)| \mid (s, \rho) \in E\} \subset \mathbb{R}\,.$$

We show below that, in fact, $\dim_H(E(\theta)) = \dim_H(\operatorname{proj}_\theta(E))$. We need the following notions: a real number $x \in \mathbb{R}$ is *computable* if there exists a machine that uniformly on input $k < \omega$ (or $\overline{k}$) outputs a rational $q \in \mathbb{Q}$ (or $\overline{q}$) such that $q \in B_{2^{-k}}(x)$ [27, Thm 5.1.2]; this naturally extends to $\mathbb{R}^m$ for $m \geq 1$.

Lemma 6.3.7. *Let* $m \geq 1$*. Every computable real* $x \in \mathbb{R}^m$ *has dimension* 0.

Proof. Suppose M with program p is a machine that on input $\overline{s}$ for $s <$

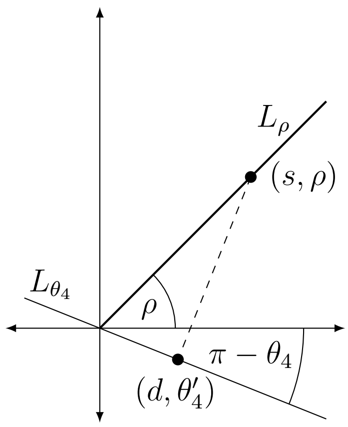

Figure 6.2: For θ_3, θ_4, a little work is needed: here, the projections meet in the fourth quadrant. Therefore, defining $\theta_4' = \pi - \theta_4$, and the definition of cos yield $d = s\cos(\rho + \pi - \theta_4) = s|\cos(\theta_4 - \rho)|$.

ω computes $\overline{q_s}$ for some $q_s \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$. Then $h\left(0^{l(p)}1p\overline{s}\right) = \overline{q_s} \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$ and so $K_s(x) \leq l(\overline{s}) + c$. Recall that $l(\overline{s}) \leq \log(s) + 1$, thus

$$\dim(x) \leq \liminf_{s \to \infty} \frac{\log(s) + 1 + c}{s} = 0. \qquad \square$$

Lemma 6.3.8. *Every countable set $E \subset \mathbb{R}^2$ has Hausdorff dimension 0.*

Proof. Suppose $E = \{x_i \,|\, i < \omega\}$, and let $X = \bigoplus \overline{x_i}$, the infinite join. Let M with program p be a machine with oracle access to X which, on input $(\bar{i}, \overline{s})$, computes $\overline{x_i}[s]$. Then it is clear that M computes all x_i, and hence by Lemma 6.3.7 and the point-to-set principle 5.3.3 we have

$$\dim_H(E) \leq \sup_{x \in E} \dim^X(x) = 0. \qquad \square$$

Lemma 6.3.9. *Let $r \in \mathbb{R}$. Then for every oracle $A \in 2^\omega$ the following hold.*

1. $\dim^A(r) = \dim^A(r, 0)$
2. $\dim^A(r) = \dim^A(-r)$

Proof. It is easily seen, modulo machine constants, that

$$K(\overline{r}[s]) \leq K(\overline{r}[s]\overline{0}[s]) \leq K(\overline{r}[s]) + K(\overline{0}[s]).$$

Since 0 is computable, Lemma 6.3.7 implies $\lim_{s\to\infty} \frac{K(\overline{0}[s])}{s} = 0$. Applying lim inf yields item 1. For item 2, observe that it is easy to compute $\overline{-r}[s]$ from $\overline{r}[s]$, which immediately implies the result. Both arguments relativise. □

Lemma 6.3.10. *Let $\theta \in [0, \pi)$. If $E \subset \mathbb{D}$ then*

$$\dim_H(\operatorname{proj}_\theta(E)) = \dim_H(E(\theta)).$$

Proof. Fix $\theta \in [0, \pi]$ and let $(s, \rho) \in \mathbb{D}$. For brevity, define $p(s, \rho)$ so that

$$p(s, \rho) = |\operatorname{proj}_\theta(s, \rho)| = s|\cos(\theta - \rho)|$$

by Lemma 6.3.6. Now item 1 of Lemma 6.3.9 implies

$$\dim^A(p(s, \rho)) = \dim^A(p(s, \rho), 0)$$

for every oracle $A \in 2^\omega$. Hence let

$$P_\theta(E) = \{(p(s, \rho), 0) \mid (s, \rho) \in E\} \subset \mathbb{R}^2\,.$$

It is now easy to see that $\dim_H(E(\theta)) = \dim_H(P_\theta(E))$ by the point-to-set principle 5.3.3.

We now appeal to Corollary 5.1.5: Hausdorff dimension is invariant under rotations. However, rotating $P_\theta(E)$ by θ anti-clockwise is not necessarily equal to $\operatorname{proj}_\theta(E)$: if there exists $(s, \rho) \in E$ for which $|\theta - \rho| > \pi/2$ then $\operatorname{proj}_\theta(s, \rho) = (p(s, \rho), \theta + \pi)$, not $(p(s, \rho), \theta)$. This is easily accounted for: whenever $(s, \rho) \in E$ and $|\theta - \rho| > \pi/2$ then, passing to Euclidean coordinates, consider $(-p(s, \rho), 0)$ instead. To this end, let

$$p^*(s, \rho) = \begin{cases} p(s, \rho) & \text{if } |\theta - \rho| \leq \pi/2 \\ -p(s, \rho) & \text{otherwise.} \end{cases}$$

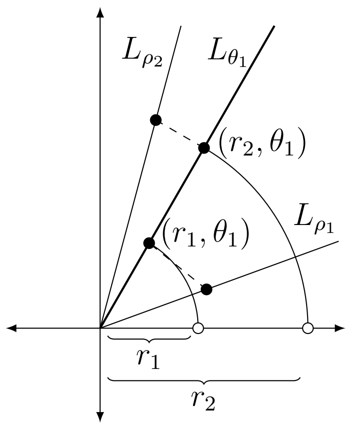

Figure 6.3: If $|\rho - \theta| \leq \pi/2$ then it suffices to consider the length of the projections on the first axis, and rotate (see ρ_1, ρ_2 and θ_1).

and hence define, in Euclidean coordinates, the set

$$P^*_\theta(E) = \{(p^*(s, \rho), 0) \mid (s, \rho) \in E\};$$

compare this with figs. 6.3 and 6.4. By items 1 and 2 of Lemma 6.3.9, it is immediate that $\dim^A(p(s, \rho), 0) = \dim^A(p^*(s, \rho), 0)$ for all $A \in 2^\omega$. The point-to-set principle implies $\dim_H(P_\theta(E)) = \dim_H(P^*_\theta(E))$. Further, rotating $P^*_\theta(E)$ by θ yields $\mathrm{proj}_\theta(E)$. Hence Corollary 5.1.5 yields

$$\dim_H(E(\theta)) = \dim_H(P_\theta(E)) = \dim_H(P^*_\theta(E)) = \dim_H(\mathrm{proj}_\theta(E)). \qquad \square$$

6.4 The Proof of Theorem 6.4.1

In this section, we construct the following counterexample: a plane set of Hausdorff dimension 1, all of whose projections have dimension 0. Using the results from section 6.3, we will argue in polar coordinates.

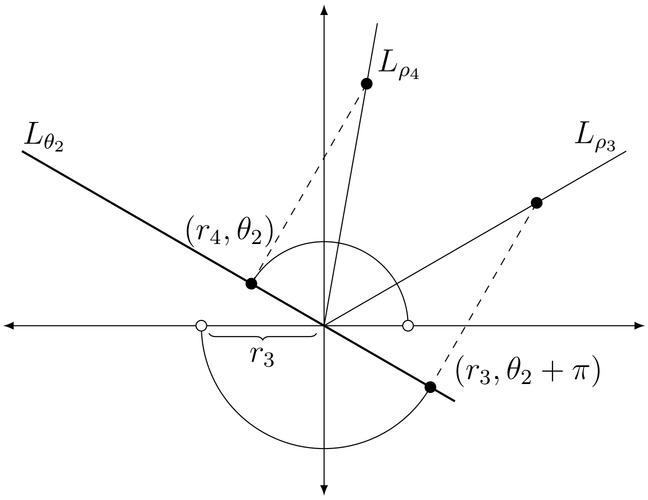

Figure 6.4: The case of ρ_4 and θ_2 is the same as in fig. 6.3. In the remaining case, we need to work a little harder: here, we must first mirror along the second axis and then rotate by θ to obtain the correct length (see θ_2 and ρ_3).

Theorem 6.4.1 (V=L)**.** *There exists a co-analytic set* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *while, for every* $\theta \in [0, \pi]$ *we have* $\dim_H(\text{proj}_\theta(E)) = 0$.

The following folklore lemma will be important in the construction.[4] We give a proof using effective dimension and the point-to-set principle 5.3.3.

Lemma 6.4.2. *If* $E \subset \mathbb{R}^2 \setminus \{0\}$ *intersects every line through the origin in* $\mathbb{D}$, *then* $\dim_H(E) \geq 1$.

Proof. Let $A \in 2^\omega$ be an oracle. There exists $B \in 2^\omega$ random relative to A. Thus $\overline{\theta} = 0001B \in 2^\omega$ codes a real $\theta \in (0, 1)$. Since B is random relative to A, we know $K^A(B \upharpoonright s) \geq s - c$ for some constant c. As $B \upharpoonright s$ is easily computable from $\overline{\theta}[s]$ we have

$$s - c \leq K^A(B \upharpoonright s) \leq K^A\left(\overline{\theta}[s]\right) + c'$$

[4]To the author's knowledge, this lemma has not yet appeared in print.

for some machine constant c'. Thus

$$\dim^A(\theta) = \liminf_{s\to\infty} \frac{K^A\left(\overline{\theta}[s]\right)}{s} \geq \liminf_{s\to\infty} \frac{K^A(B \upharpoonright s)}{s} \geq \liminf_{s\to\infty} \frac{s-c}{s} = 1.$$

Since E intersects the line L_θ, there exists $r > 0$ for which $(r, \theta) \in E$. Thus

$$\dim^A(r,\theta) = \liminf_{s\to\infty} \frac{K^A(\overline{r}[s]\overline{\theta}[s])}{s}.$$

We can easily compute $\overline{\theta}[s]$ from $\overline{r}[s]\overline{\theta}[s]$, so $K^A(\overline{\theta}[s]) \leq K^A(\overline{r}[s]\overline{\theta}[s]) + c''$ for some machine constant c''. Hence

$$\dim^A(r,\theta) = \liminf_{s\to\infty} \frac{K^A(\overline{r}[s]\overline{\theta}[s])}{s} \geq \liminf_{s\to\infty} \frac{K^A(\overline{\theta}[s])}{s} = \dim^A(\theta) = 1.$$

Since A was arbitrary, the result follows. □

6.4.1 The Roadmap Towards a Proof

We assume $V{=}L$, and hence let $B = \{\theta_\alpha \,|\, \alpha < \omega_1\}$ enumerate $[0, \pi/2]$. We want to argue by induction on ω_1 and hence build $E \subset \mathbb{D}$ satisfying Theorem 6.4.1 in stages; the angles in B are the *conditions* (or *requirements*) which need to be satisfied. During our construction, when considering condition φ, we also handle $\varphi + \pi/2$ at the same time. By Theorem 2.3.4, at stage α we have access to all points (r_i, θ_i) already enumerated into E. We aim to satisfy condition θ_α, denoted by θ for short. We argue as follows:

(1) Let $A_\alpha = \{(r_i, \theta_i) \,|\, i < \omega\}$, the set of points already enumerated into E. For each $i < \omega$ the angular coordinate θ_i tells us which condition we have already satisfied.

(2) Construct $r \in (0, 1)$ such that $\dim(r|\cos(\theta - \theta_i)|) = 0$ and $\dim(r|\cos(\theta + \pi/2 - \theta_i)|) = 0$ for all $i < \omega$. This suffices by Lemma 6.3.10.

(3) Enumerate the pair (r, θ) into E.

Observe that the set of reals in item (2) must be cofinal in the Turing degrees for Theorem 2.3.4 to apply. The following proposition is essential.

Proposition 6.4.3. *Suppose $a_i \in (0,1)$ for all $i < \omega$. Then the set of reals $\{r \in (0,1) \mid \dim(a_i r) = 0$ for all $i < \omega\}$ is cofinal in the Turing degrees.*

We will postpone the proof of Proposition 6.4.3 to section 6.4.2. However, having it in hand we may already give a proof of Theorem 6.4.1. One additional lemma is needed before we do so.

Lemma 6.4.4. *$\{x \in \mathbb{R} \mid \dim^A(x) = a\}$ is Borel for every $A \in 2^\omega, a \in [0,1]$.*

Proof. We give a proof using Theorem 2.1.20. Let $z = A \oplus a$. The function K^A is computable from z', hence so is the sequence $(q_n) = \left(\frac{K^A(\overline{x}[n])}{n}\right)$. The first bit after the binary point[5] of $\liminf_{n\to\infty} q_n$ is 0 if

$$(\forall m)(\exists n \geq m)\left(q_n < \frac{1}{2}\right)$$

which is computable from $z^{(3)}$; the other bits are computed similarly. Equality with a is computable from another jump, hence four jumps suffice. □

Proof of Theorem 6.4.1. We define $F \subset \mathbb{D}^{\leq\omega} \times [0,\pi/2] \times \mathbb{D}$ such that

$$(A, \varphi, (r,\theta)) \in F \text{ if and only if}$$
$$\varphi = \theta \text{ and for all } (r',\theta') \in \operatorname{ran}(A) \text{ we have}$$
$$\dim(r|\cos(\varphi - \theta')|) = \dim(r|\cos(\varphi + \pi/2 - \theta')|) = 0.$$

In particular, observe that every point witnessing that condition φ is satisfied lies on the line L_φ. In order to apply Theorem 2.3.4, we must show that F is co-analytic; but this follows immediately from Lemma 6.4.4. Hence let $\varphi \in [0,\pi/2]$. By definition, given $\alpha < \omega_1$ we have

[5] Recall that $\dim(x) \in [0,1]$ for all $x \in \mathbb{R}$.

$$F(A, \varphi) = \{(r, \theta) \mid (A, \varphi, (r, \theta)) \in F\}.$$

Suppose $A = \{(r_i, \theta_i) \mid i < \omega\} \in \mathbb{D}^{\leq\omega}$, and hence countable. Let

$$a_i = |\cos(\varphi - \theta_i)| \text{ and } b_i = |\cos(\varphi + \pi/2 - \theta_i)|.$$

Observe that, by construction, we have $(r, \theta) \in F(A, \varphi)$ if and only if $\theta = \varphi$ and $\dim(ra_i) = \dim(rb_i) = 0$ for all $i < \omega$. Now Proposition 6.4.3 implies that this section is cofinal in the Turing degrees. Therefore, using Lemma 6.4.4, we see that Theorem 2.3.4 is applicable: there exists a co-analytic set

$$E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\} \subset \mathbb{R}^2$$

which is compatible with F. In particular, there exist enumerations $\{\varphi_\alpha \mid \alpha < \omega_1\} = [0, \pi/2]$ and $\{A_\alpha \mid \alpha < \omega_1\}$ of $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\} = E \restriction \alpha$ such that

$$(r_\alpha, \theta_\alpha) \in F(A_\alpha, \varphi_\alpha).$$

for each $\alpha < \omega_1$. In particular, $\theta_\alpha = \varphi_\alpha$.

For the verification, let $\varphi \in [0, \pi]$. We show that $\dim_H(\text{proj}_\varphi(E)) = 0$. By Lemma 6.3.10, it suffices to show that $\dim_H(E(\varphi)) = 0$, where $E(\varphi) = \{r|\cos(\varphi - \theta)| \mid (r, \theta) \in E\}$. Observe that either $\varphi = \varphi_\delta \in [0, \pi/2]$ for some $\delta < \omega_1$; or $\varphi = \varphi_\delta + \pi/2 \in (\pi/2, \pi]$ for some $\varphi_\delta \in [0, \pi/2]$. Let δ be such, and recall that $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$. We consider the points enumerated before and those enumerated after condition φ_δ separately.

$\leq \delta$: At condition φ_δ, define (analogous to Lemma 6.3.8) the oracle

$$X = \bigoplus \left\{ \overline{r_\beta|\cos(\varphi_\delta - \theta_\beta)|}, \overline{r_\beta|\cos(\varphi_\delta + \pi/2 - \theta_\beta)|} \,\middle|\, \beta \leq \delta \right\}.$$

Then X computes $r_\beta|\cos(\varphi_\delta - \theta_\beta)|$ and $r_\beta|\cos(\varphi_\delta + \pi/2 - \theta_\beta)|$ for all $\beta \leq \delta$. Since either $\varphi = \varphi_\delta$ or $\varphi = \varphi_\delta + \pi/2$, Lemma 6.3.7 implies in particular that for all $\beta \leq \delta$

$$\dim^X(r_\beta|\cos(\varphi - \theta_\beta)|) = 0.$$

$> \delta$: We show that for every $\beta > \delta$ we have $\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0$. Let $\delta < \beta < \omega_1$. Then $(r_\beta, \theta_\beta) \in F(A_\beta, \varphi_\beta) = F(E \restriction \beta, \varphi_\beta)$. But the conditions we have already attended to at stage β are exactly the angular coordinates of the points enumerated into $E \restriction \beta$; in particular, $E \restriction \beta = \{(r_\alpha, \varphi_\alpha) \mid \alpha < \beta\}$. So for all $\gamma < \beta$, again by definition of F, we have

$$\dim(r_\beta|\cos(\varphi_\gamma - \theta_\beta)|) = \dim(r_\beta|\cos(\varphi_\gamma + \pi/2 - \theta_\beta)|) = 0.$$

Since $\delta < \beta$ and either $\varphi = \varphi_\delta$ or $\varphi = \varphi_\delta + \pi/2$ we have in particular

$$\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0.$$

We picked $\delta < \beta < \omega_1$ arbitrarily, hence this holds for all such β.

Thus, by the point-to set principle 5.3.3 and Lemma 6.3.10, we have

$$\begin{aligned}
\dim_H(\mathrm{proj}_\varphi(E)) &= \dim(E(\varphi)) \\
&= \min_{A \in 2^\omega} \sup_{\alpha < \omega_1} \dim^A(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\
&\leq \dim^X(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\
&= 0.
\end{aligned}$$

Now $\dim_H(E) \geq 1$ by Lemma 6.4.2. □

The fact that $\dim_H(E) = 1$ is a consequence of the following corollary.

Corollary 6.4.5. *Suppose $E \subset \mathbb{D}$. Then $\dim_H(\mathrm{proj}_\theta(E)) \geq \dim_H(E) - 1$.*

Proof. Suppose $(r, \theta) \in \mathrm{proj}_\theta(E)$. By Lemma 6.3.6, $r = s|\cos(\theta - \rho)|$ for some $(s, \rho) \in E$. Thus there is only one piece of information missing: given (r, θ), we may compute s from ρ, and vice versa. Hence suppose $\dim(r, \theta) = \epsilon$. Since $\dim(s), \dim(\rho) \leq 1$ we see that $\dim(s, \rho) \leq \dim(r, \theta) + 1$, as desired. □

6.4.2 Proving Proposition 6.4.3

An interval is *(open) dyadic* if it is of the form $(j/2^k, (j+1)/2^k)$. Intervals of the form $[j/2^k, (j+1)/2^k]$ are *closed dyadic*. Observe that if $x \in (j/2^k, (j+1)/2^k)$ then $|x - j/2^k| \leq 2^{-k}$, and hence x and $j/2^k$ agree on the first k bits in their binary expansion: both start with the binary expansion of j.

In the results below, we work with open intervals in $(0, 1)$. All reals are expressed in binary. Instead of manipulating intervals directly, we will argue in terms of dyadic reals, which we will express by their finite binary expansion. For this, we introduce the following notation. If $\sigma \in 2^{\leq\omega}$, let $\tilde{\sigma} = 0.\sigma \in \mathbb{R}$. If $\sigma \in 2^{<\omega}$, let $\tilde{\sigma}_+ = 0.\sigma 1^\infty \in \mathbb{R}$; let $[\tilde{\sigma}]$ denote the open interval $(\tilde{\sigma}, \tilde{\sigma}_+)$. If $a \in \mathbb{R}$ then $a[\tilde{\sigma}] = (a\tilde{\sigma}, a\tilde{\sigma}_+)$.

Some basic facts that follow directly from our definitions are the following:

- If $\sigma \in 2^{<\omega}$ then $[\tilde{\sigma}]$ is a dyadic interval; so if $x \in [\tilde{\sigma}]$ then x and $\tilde{\sigma}$ agree on the initial segment of length $l(\sigma)$. We can think of x *extending* $\tilde{\sigma}$.
- Conversely, if I is dyadic and $\sigma \in 2^{<\omega}$ is such that $\tilde{\sigma}$ is the left-end point of I, then $I = [\tilde{\sigma}]$.
- If I is dyadic and $\sigma \in 2^{<\omega}$ is such that $\tilde{\sigma} \in I$ then $[\tilde{\sigma}] \subset I$.
- In particular, if $\sigma, \rho \in 2^{<\omega}$ then $\sigma \prec \rho$ if and only if $\tilde{\rho} \in [\tilde{\sigma}]$.

Lemma 6.4.6. *Let $\sigma \in 2^{<\omega}$ and $a \in (0, 1)$. Suppose $0 < \epsilon < 1$. There exist strings $\rho, \tau \in 2^{<\omega}$ such that:*

1. $\sigma \prec \rho$
2. $a[\tilde{\rho}] \subset [\tilde{\tau}]$
3. $K(\tau)/l(\tau) < \epsilon$

Proof. Let σ, a and ϵ be given. Consider $a[\tilde{\sigma}]$. Since $[\tilde{\sigma}]$ is open, so is $a[\tilde{\sigma}]$, and thus it contains a closed dyadic interval. Take the largest (in diameter)

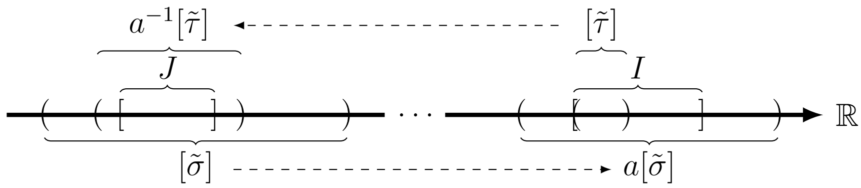

Figure 6.5: We start on the left and argue anti-clockwise: considering $a[\tilde{\sigma}]$ yields an open interval; the largest closed dyadic interval inside is I. A suitable $\tilde{\tau} \in I$ yields $a^{-1}[\tilde{\tau}]$. The largest dyadic interval contained in it is J with left end-point $d = \tilde{\rho}$. Hence $[\tilde{\rho}] \subset J$, where the interior of J equals $[\tilde{\rho}]$.

such interval I, and pick $\tau' \in 2^{<\omega}$ such that $\tilde{\tau}'$ is the left end-point of I. By closedness, $\tilde{\tau}' \in a[\tilde{\sigma}]$. By standard results on Kolmogorov complexity, there exists a smallest $s < \omega$ such that $\tau = \tau' 0^s$ satisfies

$$\frac{K(\tau)}{l(\tau)} < \epsilon.$$

In particular, $\tilde{\tau}' = \tilde{\tau} \in I$. Consider $[\tilde{\tau}]$, which is open, and hence so is $a^{-1}[\tilde{\tau}]$. Let J be the largest closed dyadic interval contained in $a^{-1}[\tilde{\tau}]$, and call its left end-point d. Again by closedness, $d \in a^{-1}[\tilde{\tau}]$. Let $\rho \in 2^{<\omega}$ be such that $\tilde{\rho} = d$. Now $\sigma \prec \rho$: by construction, $\tilde{\rho} = d \in J \subset a^{-1}[\tilde{\tau}]$. The string τ properly extends τ', thus $[\tilde{\tau}] \subset [\tilde{\tau}']$. Since $\tilde{\tau}'$ is the left end-point of I, the interior of I equals $[\tilde{\tau}']$. Hence $[\tilde{\tau}] \subset [\tilde{\tau}'] \subset I \subset a[\tilde{\sigma}]$, and so $\tilde{\rho} \in a^{-1}[\tilde{\tau}] \subset a^{-1}(a[\tilde{\sigma}]) = [\tilde{\sigma}]$. Further, $a[\tilde{\rho}] \subset [\tilde{\tau}]$, since $[\tilde{\rho}] \subset J \subset a^{-1}[\tilde{\tau}]$ with all inclusions proper. □

In order to achieve cofinality in the Turing degrees when constructing a suitable $r \in (0, 1)$, we need to satisfy each condition (as per item (2) in section 6.4.1) while coding a given oracle $A \in 2^{\omega}$ into r. Let

$$\nu(k) = 2^{2^k}$$

determine at which bits of r to code A. We will use the gaps in between the range of ν to satisfy the conditions. We call ν the **folding map**.

The construction of r

Suppose (a_i) is the set of conditions, where $a_i \in (0,1)$ for all $i < \omega$. We construct $r \in (0,1)$ in stages, by determining its binary expansion, which is given by successive extensions $x_0 \prec x_1 \prec x_2 \prec \dots$ with $x_i \in 2^{<\omega}$. We argue by induction on ω.

(1) Let $A \in 2^\omega$ be given.

(2) Let $x_0 = \emptyset$, the empty string.

(3) Let x_k be given. At stage $k+1$, decode $k+1 = \langle i, n\rangle$ via Cantor's pairing function, for instance, and attend to requirement i. Hence we attend to each requirement infinitely often.

(4) Apply Lemma 6.4.6 with $a = a_i$ and $\epsilon = \frac{1}{k}$ to find an extension $\rho_k \succ x_k$.

(5) Let $t = \nu(k+1) - l(\rho_k) - 1$ and $d = A(k)$ and define

$$x_{k+1} = \begin{cases} \rho_k 0^t d & \text{if } l(\rho_k) < \nu(k+1) \\ (\rho_k \restriction (\nu(k+1)-1))d & \text{otherwise.} \end{cases}$$

Therefore, if $k > 0$ then $l(x_k) = \nu(k)$ by induction.

(6) Define $x = \bigcup_{k<\omega} x_k$, and hence let $r = \tilde{x}$.

(7) Observe that A is computably *folded into* x: for all $k < \omega$, we have $x(\nu(k+1)-1) = A(k)$.

In order to complete the proof of Proposition 6.4.3, we need to ensure that the second case in the equation in item (5) only occurs finitely often for each requirement a_i. The next lemma ensures this. Before we proceed with the proof, a few useful facts about intervals follow. Let $(x,y) \subset (0,1)$.

(i) By $\mathrm{diam}((x,y)) = y - x$ we denote the diameter of (x,y). If $\sigma \in 2^{<\omega}$ then $\mathrm{diam}([\tilde{\sigma}]) = 2^{-l(\sigma)}$. In particular, $-\log(\mathrm{diam}([\tilde{\sigma}])) = l(\sigma)$.

(ii) If $k < \omega$ is such that $k \geq -\log(\mathrm{diam}((x,y)))+2$ then there exists $j < \omega$ such that the closed dyadic interval $[j/2^k, (j+1)/2^k] \subset (x,y)$.

Lemma 6.4.7. *For each $a_i \in (0,1)$ there exists $M_i < \omega$ such that if $k+1 > M_i$ and $k+1 = \langle i, n\rangle$ attends to requirement a_i, then $l(\rho_k) < \nu(k+1)$.*

Proof. Fix some $a_i = a$ and suppose we are at stage $k+1 = \langle i, n\rangle$. Let $\rho = \rho_k$. Recall that $\tilde{\rho} \in J \subset a^{-1}[\tilde{\tau}]$. Observe that $\mathrm{diam}(a^{-1}[\tilde{\tau}]) = a^{-1}2^{-l(\tau)}$. Now, since J is defined to be the maximal (in diameter) closed dyadic interval inside $a^{-1}[\tilde{\tau}]$, and since $\tilde{\rho}$ is the left end-point of J, items (i) and (ii) imply

$$\begin{aligned} l(\rho) &\leq -\log(\mathrm{diam}(a^{-1}[\tilde{\tau}])) + 2 \\ &= \log(a) - \log\left(2^{-l(\tau)}\right) + 2 = \log(a) + l(\tau) + 2. \end{aligned}$$

Recall that $\tau = \tau' 0^s$, and hence

$$l(\rho) \leq \log(a) + l(\tau') + s + 2.$$

Recall that ρ is an extension of x_k (so $x_k = \sigma$ in Lemma 6.4.6). By construction, $\tilde{\tau}' \in I \subset a[\tilde{x}_k]$, where I is dyadic maximal in $a[\tilde{x}_k]$. Therefore

$$l(\tau') \leq -\log(\mathrm{diam}(a[\tilde{x}_k])) + 2 = -\log(a) + l(x_k) + 2$$

from which we obtain via item (5) that

$$l(\rho) \leq l(x_k) + s + 4 = \nu(k) + s + 4$$

Recall that we are currently at stage $k+1$, and we set out to build $x_{k+1} \succ x_k$, where $l(x_{k+1}) = \nu(k+1)$. Our construction is successful if we need not truncate ρ (as in the latter case in item (5)). In such a case, $l(\rho) < l(x_{k+1}) = \nu(k+1)$. Hence it suffices to show that, eventually, $s < \nu(k+1) - \nu(k) - 4$.

Recall that s is chosen so that $\frac{K(\tau)}{l(\tau)} = \frac{K(\tau' 0^s)}{l(\tau')+s} < \frac{1}{k}$. Simplify this as follows:

$$\begin{aligned}\frac{K(\tau' 0^s)}{l(\tau')+s} &\leq \frac{K(\tau') + K(0^s) + c'}{s} \\ &\leq \frac{K(\tau')}{s} + \frac{K(s)}{s} + \frac{c''}{s} \\ &\leq \frac{l(\tau') + 2\log(l(\tau'))}{s} + \frac{\log(s) + 2\log(\log(s)+1)}{s} + \frac{c}{s}\end{aligned}$$

for a sum of machine constants c. These terms are easily bounded. Clearly, $\frac{c}{s} < \frac{1}{3k}$ if $s > 3kc$. For the middle term, observe that if $s \geq 2$ then $\log(s) + 2\log(\log(s)+1) < 3\log(s)$. Hence,

$$\frac{\log(s) + 2\log(\log(s)+1)}{s} < \frac{3\log(s)}{s}.$$

Since $\log(s)/s$ is monotonically decreasing, if $s > 2^k$ then

$$\frac{3\log(s)}{s} < \frac{3\log\left(2^k\right)}{2^k} = \frac{3k}{2^k}.$$

Then $\frac{3k}{2^k} < \frac{1}{3k}$ if $9k^2 < 2^k$ which holds for $k \geq 10$. Hence, for large enough k, the bound $s > 2^k$ suffices.

For the first term, recall that $l(\tau') \leq -\log(a) + \nu(k) + 2$. Since $a \in (0,1)$ we know $-\log(a) > 0$. So, for large enough k, it follows that

$$\begin{aligned}\frac{l(\tau') + 2\log(l(\tau'))}{s} &\leq \frac{-\log(a) + \nu(k) + 2 + 2\log(-\log(a) + \nu(k) + 2)}{s} \\ &\leq \frac{-\log(a) + 3\nu(k)}{s}\end{aligned}$$

Since a is fixed we have, for large enough k, that

$$\frac{l(\tau') + 2\log(l(\tau'))}{s} \leq \frac{-\log(a) + 3\nu(k)}{s} \leq \frac{4\nu(k)}{s}.$$

Now observe that $\frac{4\nu(k)}{s} \leq \frac{1}{3k}$ if $s > 12k\nu(k)$. Choosing a large enough k we hence see that $s > \max\left\{3kc, 2^k, 12k\nu(k)\right\}$ suffices, which, again, reduces to

$s > 12k\nu(k)$ once k is large enough. Finally, note that $12k\nu(k)+1 < \nu(k+1) - \nu(k) - 4$ for $k \geq 3$. Thus, once k is sufficiently large to satisfy all conditions above, $s = 12k\nu(k) + 1$ satisfies $\frac{K(\tau' 0^s)}{s} < \frac{1}{k}$ while $s < \nu(k+1) - \nu(k) - 4$. So, eventually, $l(\rho)$ is small enough. □

Proof of Proposition 6.4.3. Let $A \in 2^\omega$ be given, and suppose (a_i) is the countable sequence of requirements. Construct $x = \bigcup_{k<\omega} x_k$ as in section 6.4.2. Let $r = \tilde{x}$. From section 6.2 and Definition 2.3.3 it is easily seen that A can be obtained computably from the binary expansion of r. Hence we only need to show that the dimension of $a_i r$ is minimal. Fix $i < \omega$ and consider a_i. By Lemma 6.4.7 there exists M such that if $k > M$ and $k = \langle i, n \rangle$ then $\rho_k \prec x$. For each such k, let τ_k be as obtained from Lemma 6.4.6 alongside ρ_k. Now

$$\dim(a_i r) = \liminf_{s\to\infty} \frac{K(\overline{a_i r}[s])}{s}$$

by Corollary 6.2.2. By construction, $a_i[\tilde{\rho}_k] \subset [\tilde{\tau}_k]$ and $a_i \tilde{\rho}_k \in [\tilde{\tau}_k]$. Thus $a_i r = a_i \tilde{x} \in [\tilde{\tau}_k]$. Further, $K(\tau_k)/l(\tau_k) < 1/k$. Let $D = \{k > M \mid k = \langle i, n \rangle \text{ for some } n\}$. Then

$$\dim(a_i r) \leq \liminf_{k\to\infty,\, k\in D} \frac{K(\overline{a_i r}[l(\tau_k)])}{l(\tau_k)} \leq \liminf_{k\to\infty,\, k\in D} \frac{K(\tau_k) + c}{l(\tau_k)} \leq \liminf_{k\to\infty,\, k\in D} \frac{1}{k} = 0$$

where c is the machine constant obtaining τ_k from $\overline{a_i r}[l(\tau_k)]$. □

6.5 The Proof of Theorem 6.5.1

Theorem 6.5.1. *For every $0 < \epsilon < 1$, there exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1 + \epsilon$ while, for every $\theta \in [0, 2\pi)$ we have*

$$\dim_H(\mathrm{proj}_\theta(E)) = \epsilon.$$

Observe that $\epsilon = 0$ in Theorem 6.5.1 recovers Theorem 6.4.1. Further, the case $\epsilon = 1$ is trivial: if $E \subset \mathbb{R}^2$ satisfies $\dim_H(E) = 2$ then Corollary 6.4.5 implies $1 \leq \dim_H(\mathrm{proj}_\theta(E)) \leq 1$ for each θ. Hence our theorems exhaust all cases. Theorem 6.5.1 is optimal by Corollary 6.4.5.

6.5.1 Another Roadmap Towards a Proof

Let $0 < \epsilon < 1$. Assuming $V{=}L$, we argue as follows.

1. Fix an enumeration $\{\varphi_\alpha \,|\, \alpha < \omega_1\}$ of $[0, \pi/2]$.
2. At stage α, let $A_\alpha = \{(r_i, \theta_i) \,|\, i < \omega\}$, the set of all points already enumerated into our set.
3. Let $X \in 2^\omega$ be the sequence whose bits are made up of the binary expansion of φ_α. In particular, X is $\overline{\varphi}_\alpha$ with its first four bits removed (cf. section 6.2).
4. We will *not* satisfy condition φ_α by enumerating a point on L_{φ_α} into our set. Instead, we recover the already satisfied conditions by first coding them into r using a suitable folding map: if (r_i, θ_i) was enumerated into our set at stage β, then φ_β is folded into r_i, and can hence be recovered computably. Let $\{\varphi_i \,|\, i < \omega\}$ be the set of already satisfied conditions.
5. Pick $\theta \in [0, \pi/2]$ such that $\overline{\theta}$ is random relative to X.
6. Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$. Let Y be the join of X and all $\overline{a}_i$. We also assume that Y computes ϵ.
7. Construct $r \in (0, 1)$ such that:
   (a) the binary expansion of φ_α is folded computably into the binary expansion of r;
   (b) $\dim(ra_i) = \epsilon$ for all $i < \omega$;
   (c) $\dim^{Y,\overline{\theta}}(r) = \epsilon$.

8. Enumerate the pair (r, θ) into E.

Remark. At first, it might appear difficult how to control items 7b and 7c. In practice, construct r so that $\dim(ra_i) \leq \epsilon$ and $\dim^{Y,\overline{\theta}}(r) \geq \epsilon$. Equality then follows immediately from Corollary 6.4.5.

We will give some insight into the verification. Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$. In our construction of a suitable r, we adapt the methods used in the proof of Theorem 6.4.1. However, instead of inserting long strings of zeroes into the binary expansions of ra_i, we pick a suitable oracle $T \in 2^\omega$ and fold it into ra_i. The oracle T is **suitable** if it is random relative to $Y \oplus \overline{\theta}$ (and hence all $\overline{a}_i$). Now suppose r is as constructed. Then Y (which computes all $\overline{a}_i$) can compute an initial segment of T from an initial segment of r: just compute an initial segment of ra_i for the correct i. Since T is random relative to $Y \oplus \overline{\theta}$, we can force $\dim^{Y,\overline{\theta}}(r)$ to not dip too low by coding T not too sparsely.

The details can be found in section 6.5.5, and Corollary 6.4.5 then proves the result. We use the following lemma implied by *symmetry of information*.[6]

Lemma 6.5.2 ([91, Cor 13 + 15])**.** *Let $A \in 2^\omega$ be an oracle. For any $x, y \in \mathbb{R}$ we have* $\dim^A(x, y) \geq \dim^A(x) + \dim^{A,\overline{x}}(y)$.

Since Y computes X, $\dim^X(r) \geq \dim^Y(r)$ for all r. Therefore

$$\dim^X(r, \theta) \geq \dim^X(\theta) + \dim^{X,\overline{\theta}}(r) \geq \dim^X(\theta) + \dim^{Y,\overline{\theta}}(r) \geq 1 + \epsilon$$

since $\overline{\theta}$ is random relative to X, and by our construction of r. The final steps of this high-level verification are then as follows: suppose we construct E broadly as in the proof of Theorem 6.4.1. Using Theorem 5.3.3 we see that

[6]For an in-depth account of the interplay between relativised dimension and *conditional dimension* of elements of $\mathbb{R}^n$ see [91, 4.3, 4.4], who introduced the latter notion ibidem.

$$\dim_H(\mathrm{proj}_\theta(E)) = \dim_H(E(\theta)) \leq \epsilon$$

since allowing oracles can only decrease dimension. On the other hand, every oracle X appears through some $\varphi_\alpha \in [0, \pi/2]$. Hence there exists a point $(r_\alpha, \theta_\alpha)$ for which $\overline{\theta}_\alpha$ is random relative to X. Since such a point exists for *every* oracle, Theorem 5.3.3 implies

$$\dim_H(E) \geq \dim^X(r_\alpha, \theta_\alpha) \geq \dim^X(\theta_\alpha) + \dim^{X,\overline{\theta}_\alpha}(r_\alpha) \geq 1 + \epsilon$$

by Lemma 6.5.2. Then the conclusion follows from Corollary 6.4.5.

6.5.2 Folding a Suitable Oracle Into r

Fix $\epsilon \in (0, 1)$, and let

$$Z = Y \oplus \overline{\theta}$$

recalling that Y computes all a_i. Instead of constructing $T \in 2^\omega$ random relative to Z and then coding it sparsely to obtain dimension ϵ, we use a result of Athreya et al. [3, Thm 6.5]: for every $0 \leq \alpha \leq 1$ there exists $x \in \mathbb{R}$ such that $\dim(x) = \mathrm{Dim}(x) = \alpha$, obtained precisely by sparsely coding a random sequence, interleaved with strings of zeroes. This relativises by choosing a sequence random to an oracle, and their construction shows that $\dim^Z(\overline{x}) = \dim(\overline{x}) = \epsilon$, so letting $T = \overline{x}$ for a suitable $x \in \mathbb{R}$ works.

In Theorem 6.4.1 we demanded sufficiently many consecutive zeroes to appear in the image, to push the complexity down; and in our verification, we showed that, eventually, the gap between conditions will be large enough so that enough zeroes (i.e. a sufficiently large s) can be accommodated. In the present argument, we need to be more careful as we must always be able to give a good bound on how many bits of T can be computed. Hence we fix the number of bits to be appended so that there is no "overspill":

Lemma 6.5.3. *In the argument of Lemma 6.4.6, if* $s = \nu(k+1) - \nu(k) - 5$ *then* $l(\rho_k) < \nu(k+1)$.

Proof. This follows from the proof of Lemma 6.4.7: with a, ρ, τ' as in said argument, we have

$$l(\rho_k) \leq \log(a) + l(\tau') + s + 2 \leq \nu(k) + s + 4.$$

Equate the term to $\nu(k+1)$ and demand strict inequalities to finish. □

The following corollary shows that, if we have space for s bits to encode, we can code $s - 5$ bits into the image.

Corollary 6.5.4. *In the argument of Lemma 6.4.6, with* $a \in (0,1)$*: if* $l(\rho) = m$ *and* $n > m$ *then if* $s = n - m - 5$ *we have that* $l(\rho') < n$*, where* $l(\rho')$ *is the extension of* ρ *that codes* s *bits into* $r\tilde{\rho}'$.

We choose the folding map

$$\nu(k) = 2^{2^k} + k.$$

We introduce the shift summand k so as to make sure that the gaps between $\nu(k)$ and $\nu(k+1)$ have length $2^{2^{k+1}} + k + 1 - 2^{2^k} - k = 2^{2^{k+1}} - 2^{2^k} + 1$; the last bit is reserved to code a bit of φ_α into ra_i (as per item 7a). Now, the gap we have available to extend is exactly of length

$$\nu(k+1) - \nu(k) - 1 = 2^{2^{k+1}} - 2^{2^k} = 2^{2^k}\left(2^{2^k} - 1\right) \tag{6.1}$$

which is divisible by $2^{(2^k - k)}$.

6.5.3 Coding and Saving Blocks

Naively, our argument should work as follows: at each stage, we construct a radius r that, together with a suitable angle θ, satisfies the requirement at hand. In order to preserve the high dimension of r (relative to Z) and of the points ra_i, we code segments of T into each ra_i. In particular: if a_j is attended to right after a_i, and the last bit of T coded into ra_i is $T(k)$ for some $k < \omega$, then the first bit of T coded into ra_j at that stage is $T(k+1)$. Hence, with a long enough initial segment of r, the oracle Z can compute a long initial segment of T by just picking the correct a_i (which Z computes), computing ra_i, and picking out the coded bits of T.

For the sake of exposition, suppose $T \in 2^\omega$ and consider

$$T^{(m,n)} = \langle T(m), T(m+1), \ldots, T(n-1) \rangle.$$

In particular, observe that $l(T^{(m,n)}) = n - m$, and that $T(n)$ does not appear in $T^{(m,n)}$. Now, the dimension of ra_i is bounded above by the dimension of T: taking a sufficiently long initial segment $\overline{ra_i}[t]$ of ra_i, we easily find a long string of the form $T^{(m,n)}$ coded into it. Provided that $l(T^{(m,n)}) = n - m$ is large enough compared to t, this will force the dimension down—this latter condition is easily ensured by choosing a sparse enough folding map.

However, it is now difficult to show that the dimension of ra_i does not drop properly below the dimension of T. The problem is that it is in general hard to tell how many bits in the multiplication of reals are determined by a single bit: e.g. if $a = 1/\pi$ and $\tilde{\sigma} = 0.\sigma$ for some $\sigma \in 2^{<\omega}$, and $\tau \succ \sigma$, there is no bound on how many bits of the product $a\tilde{\tau}$ are correct in the sense that every extension yields the same initial segment.[7]

[7]For an extreme example, take a and σ so that $a\tilde{\sigma} = \frac{1}{2} - \epsilon$ for small ϵ, and $\operatorname{diam}(a[\tilde{\sigma}]) >$

We circumvent this issue as follows: as we extend r, we *save blocks* of bits that are coded into ra_i throughout the stage. We do this by pulling back the interval, as per Lemma 6.4.6. Hence we define the *block map* $\mu\colon \omega \to \omega$ by

$$\mu(k) = 2^{(2^k - k)}.$$

Recall that our folding map is given by $\nu(k) = 2^{2^k} + k$. Hence, at stage k with r_k in hand, we have $\nu(k+1) - \nu(k)$ many bits to extend r_k. In particular, the *number of blocks fitting into the gap of stage* $k+1$ is given by

$$\xi(k) = \frac{\nu(k+1) - \nu(k) - 1}{\mu(k)} = \frac{2^{2^k}\left(2^{2^k} - 1\right)}{2^{(2^k-k)}} = 2^k\left(2^{2^k} - 1\right). \tag{6.2}$$

Note that, by our choices, we have $\xi(k)\mu(k) = \nu(k+1) - \nu(k) - 1$.

A few lemmas are needed. Firstly, we need to have a good bound on how many bits we can code into ra_i at each stage k, and in each block. And secondly, it is not clear that saving blocks does not cost too many bits. The first is not an issue due to Corollary 6.5.4. We resolve the second later in the *cost lemma* 6.5.5, after introducing the construction in detail.

As we code T in blocks, it is prudent to describe a suitable partitioning of T beforehand: by recursion reconstruct T into segments T_k^j, where k denotes the last completed stage (so if we see T_k^j then we are in stage $k+1$), and j the active block. In summary, at stage $k+1$:

- we code $\xi(k) = 2^k\left(2^{2^k} - 1\right)$-many blocks, which follows from eq. (6.2);
- and each block of T coded into the image has length $\mu(k) - 5$, as we lose 5 bits each time as per Corollary 6.5.4.

Hence we obtain

$\frac{1}{4}$. Then each bit of precision of σ shifts the interval rightwards and halves it. So the precision of σ needed to determine the first bit (i.e. until $\frac{1}{2} \notin a[\tilde{\sigma}]$) depends on ϵ.

$$T = \bigcup_{3\leq k<\omega} \left(\bigcup_{1\leq j\leq \xi(k)} T_k^j \right)$$

where the union operator denotes concatenation. Hence

$$T = T_3^1 \cup T_3^2 \cup \ldots \cup T_3^{2040} \cup T_3^1 \cup \ldots T_k^{\xi(k)} \cup T_{k+1}^1 \ldots$$

since $\xi(3) = 2040$. As before, we lose 5 bits each time we code a T-block, so

$$l(T_k^j) = \mu(k) - 5 = 2^{(2^k-k)} - 5.$$

This confirms why the outer union starts at $k = 3$: below, we have $2^{(2^2-2)} - 5 < 0$, so there is no space to code any bits. In particular, the first stage at which bits are coded is stage $k+1 = 4$, with $l(T_k^j) = l(T_3^j) = 27$ and $\xi(k) = \xi(3) = 2040$.

6.5.4 The Construction

Recall that our folding and block map are $\nu(k) = 2^{2^k} + k$ and $\mu(k) = 2^{(2^k-k)}$, respectively. Now, the radius r is constructed as follows: suppose φ_α is the active requirement.

1. Let $A \in 2^\omega$.
2. Let $x_0 = \emptyset$, the empty string.
3. Let x_k be given. At stage $k+1$, decode $k+1 = \langle i, n\rangle$; we now attend to requirement i.
4. We iterate over all $\xi(k)$-many blocks. Let $0 \leq j < \xi(k) = 2^k(2^{2^k} - 1)$.
   (a) Let $x_k^0 = x_k$.
   (b) At block $j+1$, suppose we have x_k^j. We apply Lemma 6.4.6, but instead of coding zeroes, we code T_k^{j+1} into $a_i\tilde{x}_k^j$. Let ρ_k^{j+1} be the resulting extension. By filling up with s-many zeroes (courtesy of Lemma 6.5.3), we hence find $x_k^{j+1} = \rho_k^{j+1}0^s$ of length

$$l(x_k^j) + \mu(k) = l(x_k) + 2^{(2^k - k)}(j+1).$$

5. After the last block, we have one bit left to code A or $\overline{\varphi}_\alpha$ (this follows from eq. (6.1)). By construction, $l\left(x_k^{\xi(k)}\right) = \nu(k+1) - 1$; hence define

$$x_{k+1} = x_k^{\xi(k)} d$$

where

$$d = \begin{cases} A(k/2) & \text{if } k \text{ is even} \\ \overline{\varphi}_\alpha((k-1)/2) & \text{if } k \text{ is odd;} \end{cases}$$

hence $l(x_{k+1}) = \nu(k+1)$, as intended. Further, we code the active line into the real we are building, so that we can recover it later.

Of course, we code A as in Theorem 6.4.1 in order to apply Theorem 2.3.4.

6.5.5 The Verification

In the present context, we have two results to prove: that $\dim(ra_i) \leq \epsilon$ and that $\dim^Z(r) \geq \epsilon$, where $Z = Y \oplus \overline{\theta}$. Then the theorem follows from Corollary 6.4.5. We prove both results individually.

The dimension of ra_i

Both verification arguments are "bit counting" arguments: we exhibit a piece of a complicated string coded inside ra_i, and show that said segment is long enough in a precise sense: its length dwarves the length of all non-coded bits. Let $a = a_i$. Consider $\overline{ar}[m]$ for some m such that

$$\overline{ar}[m] = \sigma \cup \left(\bigcup_{1 \leq j \leq \xi(k)} \sigma_j T_k^j \right)$$

for some k; hence stage $k+1$ has just been completed. (Considering the strings at the end of stages is prudent as we easily have access to a long consecutive segment of T, albeit interrupted). We also know that $l(\sigma) \leq -\log(a) + l(x_k) + 2 = -\log(a) + \nu(k) + 2$, by Lemma 6.4.7. Further, the *cost* of saving a block is given by a bound on the length of each σ_j:

Lemma 6.5.5 (The cost lemma). *Let $a \in (0,1)$ and $r_m \in 2^{<\omega}$. As in Lemma 6.4.6, find $\tilde{\tau}_m$ and I_m dyadic such that $[\tilde{\tau}_m] \subset I_m \subset a[\tilde{r}_m]$; let τ'_m be the left end-point of I_m. Further, let $J \subset a^{-1}[\tilde{\tau}_m]$ be dyadic, where $\tilde{\rho}_k$ is the left-endpoint of J. Let $r_{m+1} = \rho_m 0^t$ so that $l(r_{m+1}) = l(r_m) + \mu(k)$ where k denotes the current stage. Finally, suppose τ'_{m+1} is the left end-point of $I_{m+1} \subset a[\tilde{r}_{m+1}]$. Then $|l(\tau'_{m+1}) - l(\tau_m)| \leq 7$.*

A few comments are in order. Firstly, consulting fig. 6.5 alongside the statement and proof of the above lemma is useful, as the figure serves as its motivation. Conceptually, the hypotheses of this lemma are the intermediate step between moving from one block to the next within a given stage in our construction: r_m is the available string in block m inside some stage, and $\rho \succ \sigma$ is its computed extension. Importantly, $a[\tilde{r}_{m+1}]$ contains τ_m as a substring. We ask: after saving τ_m in $a[\tilde{r}_{m+1}]$, how many bits are lost before we begin coding the next block? In particular, if we construct a real r by such approximations r_m and we have established that

$$ar \succ \tau_m \lambda \tau_{m+1}$$

for some $\lambda \in 2^{<\omega}$ by successive block saving, then how long can λ be at most?

Proof. By assumption we have $[\tilde{\tau}_m] \subset I_m \subset [\tilde{r}_m]$, and so $\mathrm{diam}([\tilde{\tau}_m]) \leq \mathrm{diam}(I_m) \leq \mathrm{diam}([\tilde{r}_m])$. Applying $-\log$ and by item (i) we have

$$-\log(\mathrm{diam}(I_m)) \in [-\log(a) + l(r_k), l(\tau_k)].$$

Since $\tilde{\tau}'_k$ is the left end-point of I_k we have in particular that $l(\tau'_m) \in [-\log(a) + l(r_m), l(\tau_m)]$. We can give an even better bound: by item (ii), we see that $l(\tau'_m) \leq -\log(\mathrm{diam}(a[\tilde{r}_m])) + 2 = -\log(a) + l(r_m) + 2$, and hence

$$l(\tau'_m) \in [-\log(a) + l(r_m), -\log(a) + l(r_m) + 2].$$

By construction, at stage $k+1$ we code $\mu(k) - 5$ bits into the image for each block (we lose 5 bits each block, as per Corollary 6.5.4). Hence $l(\tau_m) = l(\tau'_m) + (\mu(k) - 5)$. Therefore, observing by construction that $l(\tau'_{m+1}) \geq l(\tau_m)$,

$$\begin{aligned}
l(\tau'_{m+1}) - l(\tau_m) &= l(\tau'_{m+1}) - l(\tau'_m) - (\mu(k) - 5) \\
&\leq -\log(a) + l(r_{m+1}) + 2 + \log(a) - l(r_m) - (\mu(k) - 5) \\
&= (l(r_{m+1}) - l(r_m)) - (\mu(k) - 5) + 2 \\
&= \mu(k) - (\mu(k) - 5) + 2 \\
&= 7
\end{aligned}$$

where we use that the block size is $\mu(k)$, and hence $l(r_{m+1}) - l(r_m) = \mu(k)$. $\square$

Hence $l(\sigma_j) \leq 7$. For simplicity, we let

$$T_k = T_k^1 \cup \ldots \cup T_k^{\xi(k)};$$

hence $l(T_k) = \xi(k)(\mu(k) - 5)$. The next lemma provides the final technical detail in this half of our verification. For simplicity of notation, let

$$S_k = \bigcup_{1 \leq j \leq \xi(k)} \sigma_j T_k^j.$$

Lemma 6.5.6. *For $k < \omega$ and $\sigma, (\sigma_j)$ as above, we have*

$$|K(T_k) - K(\sigma S_k)| \leq O(2^{2^k}).$$

Proof. This is a "bit counting" argument: the number of bits by which T_k and σS_k differ is given by $l(\sigma)+\sum_j l(\sigma_j)$. If we also know where the σ_j's are located, then we can construct each string from the other. Thus,

$$|K(T_k) - K(\sigma S_k)| \leq K(\sigma) + \sum_{1\leq j\leq \xi(k)} K(\sigma_j, m_j)$$

omitting constants, where m_j is the index at which σ_j begins inside S_k. Now, $l(\sigma_j) \leq 7$ and $l(\sigma) \leq -\log(a) + l(x_k) + 2 = -\log(a) + 2^{2^k} + k + 2$ imply

$$\begin{aligned} l(\sigma S_k) &= l(\sigma) + \sum_{1\leq j\leq \xi(k)} l(\sigma_j) + l(T_k) \\ &\leq -\log(a) + l(x_k) + 2 + 7\xi(k) + \xi(k)(\mu(k) - 5) \\ &= -\log(a) + l(x_k) + 2 + \xi(k)(\mu(k) + 2) \end{aligned}$$

since each of the $\xi(k)$-many blocks codes $\mu(k) - 5$-many bits. Observe that

$$\xi(k)\mu(k) = 2^{2^k}(2^{2^k} - 1)$$

and hence is of order $2^{2^{k+1}}$. As $m_j \leq l(S_k)$ we see that m_j is thus at most of order $2^{2^{k+1}}$. But now $K(m_j)$ is at most of order 2^{k+1}. It is now easily seen that $\sum_j K(\sigma_j, m_j)$ is of order at most $\xi(k)2^{k+1}$, which is $O(2^{2^k})$. □

We can now complete the argument: using the previous lemma, we see

$$K(\overline{ar}[m]) = K(\sigma S_k) = K(T_k) + O(2^{2^k}).$$

Further, observe that $l(T_k)$ is of order $2^{2^{k+1}}$, since $l(T_k) = \xi(k)(\mu(k) - 5)$. As before, $\lim_{k\to\infty} \frac{2^{2^k}}{2^{2^{k+1}}} = 0$, and so we may ignore terms of order at most 2^{2^k}. Hence simplify: let $\mathcal{D}$ be the set of $m < \omega$ at which requirement $a = a_i$ has just been attended to. (In other words, $\overline{ar}[m] = \sigma S_k$ for some k.) Then

$$\dim(ar) \leq \liminf_{m\in\mathcal{D}} \frac{K(\overline{ar}[m])}{m} \leq \liminf_{m\in\mathcal{D}} \frac{K(T_k)}{m} = \epsilon$$

by definition of T.

The dimension of r with respect to Z

Recall that $Z = Y \oplus \overline{\theta}$ and that Y computes all a_i. As we need to show that $\dim^Z(r) \geq \epsilon$, it does not suffice to exhibit a set of favourable elements, such as our set $\mathcal{D}$ in the previous lemma. Instead, we show we can decode enough elements of T from *any* initial segment of r. Suppose

$$\overline{r}[m] = \sigma_1 \cdots \sigma_{k+1} b_1 \cdots b_n \tau$$

where

- σ_i denotes the initial segment of r that satisfied the stage i;
- b_j denotes the substring of r that satisfied block j of stage $k+2$; and
- τ is the initial segment of the substring satisfying block $n+1$.

Hence, observe we are at stage $k+2$, and n blocks have already been satisfied.

Inside stage $k+1$, the substring T_k has been coded into ar. Using the oracle Z which computes all a_i, we can recover T_k from $\overline{ar}$. Recall that

$$l(T_k) = \xi(k)(\mu(k) - 5).$$

Since $\lim_{k\to\infty} \frac{2^{2^k}}{2^{2^{k+1}}} = \lim_{k\to\infty} \frac{1}{2^k} = 0$, the length of T_k dwarves the lengths of $T_1 + \ldots + T_{k-1}$, so it suffices to compute the blocks saved at stage $k+1$.

Secondly, the worst case to consider above is the case where $n = 0$: in that case the initial segment σ_{k+1} needs to carry enough information to survive against τ, where τ is at most of length $\mu(k+1) - 1$. This is not an issue, since $l(T_k) = \xi(k)(\mu(k) - 5)$ and

$$\lim_{k\to\infty} \frac{\mu(k+1) - 1}{\xi(k)(\mu(k) - 5)} = 0.$$

Hence, the information provided in T_k dwarves the unfinished block τ. It now suffices to show that T_k and the completely coded substrings $T^1_{k+1}, \ldots, T^n_{k+1}$ can be easily recovered from $\overline{ar}[m]$. This is similar to Lemma 6.5.6:

- take a machine that trims $\overline{r}$ to length $\nu(k+1)-1$, and denote the resultant string by ρ (this is where stage $k+1$ has just been completed);
- compute the correct projection factor $a_i = a$ for stage $k+1$ using Z (and from the Cantor pairing function);
- compute the largest dyadic interval in $a[\tilde{\rho}]$, and let d denote its left end-point. Now

$$\overline{d} = \sigma S_k \sigma'$$

  where $l(\sigma') \leq 7$, by the cost lemma 6.5.5.
- By the previous Lemma 6.5.6, we know that the complexity of isolating T_k from $\sigma S_k \sigma'$ is not significant, as required. An identical argument recovers the n blocks.

It now follows from Lemma 6.5.6 that

$$K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1}) \leq K^Z(\overline{r}[m]) + O(2^{2^k}) + O(n2^{k+2})$$

where $n < \xi(k+1)$. Thus, in particular

$$\frac{K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})}{m} \leq \frac{K^Z(\overline{r}[m])}{m} + \frac{O(2^{2^k}) + O(n2^{k+2})}{m} \tag{6.3}$$

where $m = \nu(k+1) + n\mu(k+1) + l(\tau)$ and $n < \xi(k+1)$. Next, we verify that the length of T computed in eq. (6.3) is sufficiently long:

$$|m - l(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})| = l(\tau) + \nu(k) + 1 + 5\xi(k) + 5n.$$

Now, $m = \nu(k+1) + n\mu(k+1) + l(\tau)$ and $n < \xi(k+1)$ imply

$$\frac{l(\tau) + \nu(k) + 5\xi(k) + 5n}{m} \leq \frac{l(\tau) + \nu(k) + 5(\xi(k) + \xi(k+1))}{l(\tau) + \nu(k+1) + \xi(k+1)\mu(k+1)}.$$

Applying limits as k goes to infinity shows that the term vanishes. Going back and applying lim inf to both sides of eq. (6.3) now proves that its left-hand side equals ϵ.

Finally, since m is of order $\nu(k+1) + n\mu(k+1)$, i.e. of order at least $2^{2^{k+1}}$, the right-hand side of eq. (6.3) simplifies to its first term. Putting it all together and applying lim inf we hence obtain

$$\epsilon = \liminf_{k\to\infty} \frac{K^Z(T_k \cup T^1_{k+1} \cup \ldots \cup T^n_{k+1})}{m} \leq \frac{K^Z(\overline{r}[m])}{m} = \dim^Z(r)$$

as required. Theorem 6.5.1 now follows from the same arguments as the proof of Theorem 6.4.1, and the outline we gave at the start of this section.

6.6 Further Work

The present investigation leaves a few questions open, which we address here. Firstly, Theorem 2.3.4 produces a $\utilde{\mathbf{\Pi}}^1_1$ set of self-constructible reals satisfying the recursion. It is well-known that the set of self-constructibles $\mathcal{C}_1$ is the largest thin Π^1_1 set: it does not contain a perfect subset [71, 100]. As has been pointed out by Vidnyánszky [152, Problem 5.8], it is not clear whether in general a non-thin set solving the recursion in Theorem 2.3.4 exists. Hence, the following remains also open:

Question 6.6.1. Does there exist a non-thin $\utilde{\mathbf{\Pi}}^1_1$ set failing Marstrand's Projection Theorem, under suitable set-theoretic assumptions?

Secondly, this chapter concerned Hausdorff dimension. As it turns out, there also exists a characterisation of **Packing dimension** in terms of Kolmogorov complexity, which is due to J. Lutz and N. Lutz [91, Thm 2].

Theorem 6.6.2. *Let $n < \omega$ and $E \subset \mathbb{R}^n$. Then we have the identity $\dim_P(E) = \min_{A\in 2^\omega} \sup_{x\in E} \mathrm{Dim}^A(x)$ where $\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K(\overline{x}[r])}{r}$.*

There exist bounds on the projection of subsets under $\dim_P$. However, these are less well-behaved; the best possible lower bound for $\utilde{\mathbf{\Sigma}}^1_1$ sets was isolated by John Howroyd and Kenneth Falconer [38], improving on Maarit Järvenpää's result [62]. This leaves the (admittedly rather general) question:

Question 6.6.3. What Packing dimensions exactly can be realised in projections of sets of reals?

It should be noted that our results in this chapter only construct sets of Hausdorff dimension greater than or equal to 1. It is not clear whether a set of Hausdorff dimension ϵ with $\epsilon \in (0, 1)$ can be constructed, which also fails Marstrand's theorem. In other words, we leave the following open question:

Question 6.6.4. Is there a set $E \subset \mathbb{R}^2$ and $\epsilon \in (0, 1)$ such that $\dim_H(E) = \epsilon$ while $\dim_H(\mathrm{proj}_\theta(E)) = 0$ for all angles θ?

Finally, this line of research begs the set-theoretical question of consistency. In a sharp turn away from *failing* Marstrand's theorem, one can ask:

Question 6.6.5. Is it consistent with the usual axioms of ZF that Marstrand's theorem holds for *every* set of reals?

Connections to measurability in set theory, such us Robert Solovay's seminal paper on the consistency of all sets being Lebesgue measurable [141], have shown that combinatorial properties of measure and category can be manipulated by set-theoretical tools, principally via descriptive set theory itself [68, Chapter 3], and in particular the method of forcing [119]. From H. Friedman's work it is known to be consistent that Fubini's theorem holds for *all* sets, which might yield a measure-theoretical consistency proof [44].

On the other hand, one can ask combinatorially. We thank Liang Yu for the following suggestion:

Question 6.6.6. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ prove the statement "Marstrand's theorem holds for all sets of reals"?

Using a recent theorem of Don Stull which developed the notion of *optimal oracles*—and showed that if $E \subset \mathbb{R}^2$ has optimal oracles then it has the Marstrand property [148]—the logical setting of AD appears quite auspicious; turning the existence of optimal oracles into a suitable game, and hence using AD, might lead to a combinatorial consistency proof.

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